2021-01-04

Tuesday, December 29, 2020 6:27 PM

On Wednesday 2021-01-06 @ $16^{\underline{10}}$,

Shubham Dwivedi will give an

Introduction to G_2-geometry.

Zoom details on https://walpu.ski/ (soon)

j/w A. Doan

Thm (Castelnuovo 1889)

If $C \subset \mathbb{P}^n$ is an irreducible, non-deg. curve of degree d, then

$$g(C) \leqslant \gamma(n,d) = \overset{\sim d^2}{(n-1)\binom{m}{2} + m\cdot\varepsilon}$$

$$\text{w/ } m := \left\lfloor \frac{d-1}{n-1} \right\rfloor$$

$$\varepsilon := (d-1) \,\%\, (n-1).$$

Rmk: For $n=2$

adjunction and Riemann-Roch yield the degree-genus formula

contribution of singularities

$$g = \binom{d}{2} - \underset{\geqslant 0}{\delta} = \gamma(2,d) - \delta.$$

Pf with work $\gamma(n,d)$: ‾aus

① Riemann-Roch:

$$\chi(\mathcal{O}_C(k)) = k\cdot d + 1 - g.$$

② Serre vanishing:

$$k \gg 1 \Rightarrow h^0(\mathcal{O}_C(k)) = k\cdot d - 1 - g$$

③ $$0 \to H^0(\mathcal{O}_C(k-1)) \xrightarrow{\lambda\cdot} H^0(\mathcal{O}_C(k)) \xrightarrow{(*)} \underset{\substack{\cong \\ \mathbb{C}^d}}{H^0(\mathcal{O}_{\lambda^{-1}(0)}(k))}$$

$\lambda \in H^0(\mathcal{O}_C(1))$ generic

$$\lambda^{-1}(0) = \{z_1, \dots, z_d\}$$

$$\exists\, \ell^i_j \in H^0(\mathcal{O}_C(1)) : \ \ell^i_j(z_i) = 1,\ \ell^i_j(z_j) = 0$$

$$\Rightarrow [\ell^i_1 \cdots \ell^i_{i-1} \cdot \ell^i_{i+1} \cdots \ell^i_d](z_j) = \delta_{ij}$$

$\Rightarrow$ (*) surjective for $k \geq d$

$$\Rightarrow h^0(\mathcal{O}_C(k)) \geq (k-d)d$$

④ $$k \gg 1 \Rightarrow \quad g \leq (d-k)\cdot d + kd + 1$$

$$\leq d^2 + 1.$$ ■

* * *

Can there be analogues of Castelnuovo's bound in almost complex geometry?

Thm (McDuff) (X, J) almost cx. 4-mfd, $u : (\Sigma, j) \to X$ simple J-holomorphic

$$\Rightarrow g(\Sigma) \leq \tfrac{1}{2}\left(A\cdot A + \langle c_1(X,J), A\rangle\right) + 1$$

w/ $A := u_*[\Sigma]$.

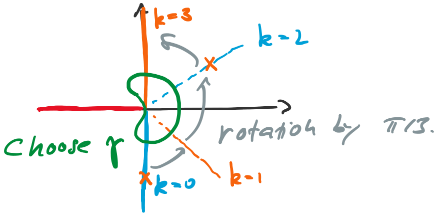
Here (and in everything that follows) it is crucial that Σ is closed. Indeed:

Thm (De Lellis – de Phillipis – Hirsch, 2019): There is a smooth oriented loop $\Gamma \subset B_1^4(0)$ passing through 0, a smooth oriented surface $\Sigma \subset B_1^4(0)\setminus\{0\}$ with infinite genus, and an almost Kähler structure (g, J) s.th. Σ is J-holomorphic and "$\partial\Sigma = \Gamma$".

Pf idea: ① Construct a holomorphic function

$$g : \mathbb{C}\setminus(-\infty, 0] \xrightarrow{\text{holo.}} \mathbb{C} \text{ with}$$

$$g^{-1}(0) = \bigcup_{k=0}^{3} \{ e^{\pi[n + i(2k-3)/6]} : n \in \mathbb{Z} \}$$



This can be done explicitly.

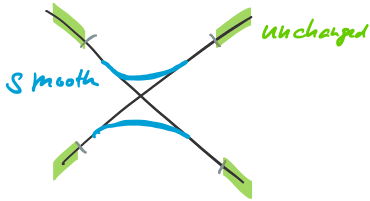
② $D :=$ disk filling γ.

$\hat{\Sigma} := \{(z^3, g(z)) : z \in D\} \subset B_1^4(0) \subset \mathbb{C}^2$

has singularities at $(\zeta^3, g(\zeta))$ for $\zeta \in g^{-1}(0)$.

modelled on transversely intersecting graphs.

④ Smooth the singularities:



Solve for $J = J_0 + \underline{\delta}$ to make this J-holomorphic. (This can be done w/ δ arbitrarily small: scaling.) ■

Gromov's h-principle implies:

Prop: (X, ω) symplectic mfd. w/ $\dim X \geq 6$,

$A \in H_2(X)$ w/ $\langle [\omega], A \rangle > 0$, $\gamma \in \mathbb{N}_0$

$\Rightarrow \exists\, J \in \mathcal{J}(\omega)$ and J-holomorphic embedding

$u : (\Sigma, j) \to X$ w/

$g(\Sigma) \geq \gamma$.

Therefore,

$$\gamma_A(X,J) := \sup\{ g \in \mathbb{N}_0 : \exists\, u : \Sigma_g \to X \text{ simple } J\text{-hol.} \}$$

might be finite for every $J \in \mathcal{J}(\omega)$, but

$$\gamma_A(X,\omega) := \sup_{J \in \mathcal{J}(\omega)} \gamma_A(X,J) = \infty.$$

index formula and transversality imply:

Prop: (X,ω) symplectic mfd. w/ $\dim X = 2m$, $J \in \mathcal{J}(\omega)$ generic, $u : (\Sigma, j) \to X$ simple J-holomorphic

$$\Rightarrow \begin{cases} \langle c_1(X,\omega), A \rangle \geq 0 & (m = 3) \\ g(\Sigma) \leq \dfrac{\langle c_1(X,\omega), A \rangle}{m-3} + 1 & (m \geq 4) \end{cases}$$

w/ $A := u_*[\Sigma]$.

This is better than Castelnuovo's bound.

What remains?

$m = 3$ and

- $\langle c_1(X,\omega), A \rangle = 0$ Calabi-Yau $\leftarrow$
- $\langle c_1(X,\omega), A \rangle > 0$. Fano

Thm (Doan–W., 2018): (X,ω) symplectic 6-mfd., $A \in H_2(X)$ w/ $\langle c_1(X,\omega), A \rangle = 0$, $J \in \mathcal{J}(\omega)$ super-

Wendl 2019

rigid ($\Leftarrow$ generic ❗)

$\Rightarrow$ $\exists$ only finitely many simple J-hol. maps representing A

$\Rightarrow$ $\gamma_A(X,J) < \infty$.

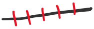
Rmk: ① The proof relies on compactness, but Gromov compactness is of no use.

② We don't have effective bounds on $\gamma_A(X,J)$.

(I suspect that effective bounds don't exist.)

③ Super-rigidity might be a red herring.

Q: Do we have



$$\sup_{t\in[0,1]} \gamma_A(X,J_t) < \infty$$

for a generic 1-parameter family $(J_t)_{t\in[0,1]}$?

Prop (Doan-W.): Yes if A is primitive.

Pf uses that the set of J for which there are simple J-hol. maps that are not embeddings has codim. 2 (and an improved cptness theorem).

④ It is natural to replace $\gamma_A(X,\omega)$ by

$$\gamma_A^h(X,\omega) := \sup\{g\geq 0 : h_{A,g}(X,\omega)\neq 0\}$$

w/ $h_{A,g}$ some "curve count".

$h = GW$ is useless b/c of multiple cover contributions.

Better: Use Gopakumar–Vafa BPS counts defined by

$$\sum_A \sum_{g\geq 0} \overset{\in \mathbb{Q}}{GW_{A,g}}\, t^{2g-2} q^A =: \sum_A \sum_{g\geq 0} \underset{\in \mathbb{Z}\ (\text{Ionel–Parker}\,!)}{BPS_{A,g}}\, t^{2g-2} \sum_{k\geq 1} \frac{1}{k}\left(\frac{\sin(kt/2)}{t/2}\right)^{2g-2} q^{kA}.$$

The Gopakumar–Vafa finiteness conjecture predicts

$$\gamma_A^{BPS}(X,\omega) < \infty.$$

Cor (Doan–W., 2019) This holds for primitive A and so does the analogue in the Fano case.

Pf: In these cases, $BPS_{A,g}$ is a count of embedded curves by Zinger's localization formula + Pandharipande's computation. ∎

Q: Are there effective bounds on γ_A^{BPS}

(in these cases)?

How to take limits of J-holomorphic curves without assuming a genus bound (a priori)?

Every $u:(\Sigma,j)\to X$ defines a linear map

$\delta_C : \Omega^2_c(X)\to\mathbb{R}$ via $\delta_C(\alpha):=\int_\Sigma u^*\alpha$.

⇝ Federer-Fleming theory of currents.

Def: ① A k-current on X is a continuous linear map $T:\Omega^k_c(X)\to\mathbb{R}$.

↑ strong C^∞ topology.

② (T_n) weakly converges to T if

$$T_n(\alpha)\to T(\alpha) \quad \text{for every } \alpha\in\Omega^k_c(X).$$

③ The boundary of T is

$$\partial T := T\circ d.$$

④ T is integer rectifiable if it can be repr. as

$$T(\alpha) = \int_A m(x)\cdot\langle\alpha(x),\vec{T}(x)\rangle\cdot d\mathcal{H}^k$$

k-dim. Hausdorff measure ↑

w/

(a) $A=\bigcup_{i=1}^{\infty} A_i \cup \mathcal{H}^k$-null set

↑ k-dim. C^1 submfds.

(b) $m : A \to \mathbb{Z}$ $\mathcal{H}^k$-measurable

(c) $\vec{T}$ $\mathcal{H}^k$-measurable section of $\Lambda^k TA$.

with $\|\vec{T}_x\| \equiv 1$.

T is integral if T and ∂T are integer rectifiable.

⑤ The support of T is the smallest closed subset $\operatorname{supp} T$ s.th $\operatorname{supp} \alpha \cap \operatorname{supp} T = \emptyset$

$$\Downarrow$$

⑥ The mass of T is $\quad T(\alpha) = 0$.

$$\mathbb{M}(T) := \sup\{T(\alpha) : \alpha \in \mathcal{D}^k_c(X) \text{ w/ } \|\alpha\| \le 1\}.$$

Thm (Federer-Fleming) Let (T_n) be a seq. of integral currents. If

$$\mathbb{M}(T_n) + \mathbb{M}(\partial T_n) \le C < \infty \text{ and } \operatorname{supp} T_n \subset K \text{ cpt.},$$

then a subseq. of (T_n) weakly converges to an integral current.

Suppose J is tamed by ω. Set

$g(v,w) := \frac{1}{2}(\omega(v, Jw) - \omega(Jv, w))$ and

$\sigma := g(J\cdot,\cdot).$ $(\neq \omega)$

The Hermitian form σ is a semi-calibration;

that is:

Wirtinger

$\|\sigma\|_g \le 1$ equivalently $\sigma|_L \le \mathrm{vol}_L$

for every or. 2-plane L.

moreover:

$$\sigma|_L = \mathrm{vol}_L \iff JL = L \text{ and cx. or.}$$

Thus

$$\mathbb{M}(\delta_C) \ge \delta_C(\sigma) = \int u^*\sigma \quad (= \text{top. if } J \text{ is } \omega\text{-compat.})$$

with "="

$$\iff \sigma_x(\vec{T}(x)) = 1 \text{ for } \mathcal{H}^2\text{-a.e. } x \in \operatorname{supp} \delta_C$$

δ_C is calibrated by σ

$\iff u$ J-hol.

Prop: Being J-hol. ($:=$ cal. by σ) is preserved by weak limits.

Deep results by De Lellis-Spadaro-Spolaor imply: every J-hol. integral current without boundary is of the form

$$\sum_{i=1}^{I} \underset{\substack{\cap \\ \mathbb{N}}}{m_i}\, \delta_{C_i}$$

4d: Taubes SW⇒Gr.

w/

simple J-hol.

$$\delta_{C_i}(\alpha) = \int_{\Sigma_i} u_i^*(\alpha).$$

With this in hand, the non-effective generic Castelnuovo bound in the CY3 case can be **proved as follows:**

Recall: $\dim X = 6$ and $\langle c_1(X,\omega), A\rangle = 0$.

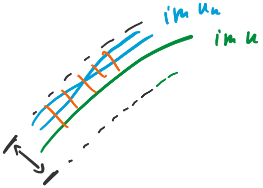
① **Wendl 2019:** a generic $J \in \mathcal{J}(\omega)$ is super-rigid, and every simple J-hol. has $\mathrm{index}(u) \geq 0$ and is an embedding if ind = 0.

② If $\gamma_A(X,J) = \infty$, there is a seq. of J-hol. embeddings $u_n : (\Sigma_n, j_n) \to X$ $\mapsto \delta_{C_n}$ w/ $(u_n)_*[\Sigma_n] = A$ and $g(\Sigma_n) \to \infty$.

Take the limit δ_C of a subseq. of (δ_{C_n}). A tiny bit of work shows that

$$\delta_C = \overset{\in \mathbb{N}}{m} \cdot \int_\Sigma u^*(-).$$

③



Suggests: u_n is multi-graph / u.

$\varepsilon_n = d(\mathrm{im}\, u_n, \mathrm{im}\, u) \to 0$

Rescale a tub. nghd of im u s.th $\varepsilon_n = 1$

and take the limit again.

rescaling limit of J

$\rightsquigarrow$ $\tilde{u}$ $\tilde{J}$-hol. map ($\neq$ zero section)

$\downarrow d$

m in normal bundle N_u.

$\lightning$ Super-rigidity. ∎

Thm (Doan–W.) Let (g_n, J_n) be a seq. of almost Hermitian str. C^∞ conv. to (g, J).

Let (C_n) be a seq. of (embedded) J_n-hol. curves,

and let C be an embedded J-hol. curve.

If (δ_{C_n}) weakly converges to $\underset{m=1}{\delta_C}$, then

(C_n) smoothly converges to C.

↑ for $n \gg 1$, $C_n = \mathrm{im}\, \exp \circ\, \xi_n$

w/ $\xi_n \in \Gamma(NC)$ and $\xi_n \xrightarrow{C^\infty} 0$.

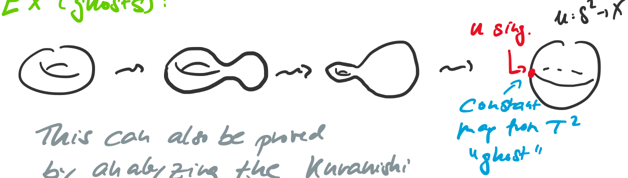
In particular, $C_n \cong C$ for $n \gg 1$.

Cor: For generic $(J_t)_{t\in[0,1]}$ and primitive A

$$\sup_{t\in[0,1]} \gamma_A(X, J_t) < \infty.$$

Cor: If $u_n \xrightarrow{\text{Gromov}} \hat{u}$, $\mathrm{im}(\hat{u})$ is smooth, and $\hat{u}$ has no multiple components, then $\hat{u}$ has no nodes and is an embedding.

Ex (ghosts):



This can also be proved by analyzing the Kuranishi models for smoothing nodes (Ionel, Zinger, DW) or a ~~top. arg. by Ekholm–Shende.~~

Pf of Thm can be achieved by using a version Allard's regularity theorem. If the C_n are known/ assumed to be smooth, then there is a simple proof using Arzelá–Ascoli and some Schauder theory.

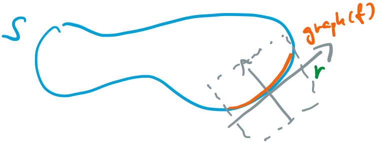
Sketch: For simplicity assume $X = \mathbb{R}^{2n}$.

① For $S \subset X$ define the C^k **regularity scale** $r_S^k : S \to [0, \infty]$ by

$r_S^k(x)$ is the supremum of those $r > 0$ for which there is an orthogonal splitting $\mathbb{R}^{2n} \cong \mathbb{R}^2 \oplus \mathbb{R}^{2n-2}$ s.th.

$$(S - x) \cap B_r^2(0) \times B_r^{2n-2}(0) = \operatorname{graph}(f)$$

with $\|f\|_{k,r} := \sum_{\ell=0}^{k} r^\ell \|\nabla^\ell f\|_\infty \leq 1$



② (S_n) **weakly C^k converges** to S if

(a) $S_n \to S$ in Hausdorff metric

(b) For every $x \in S$ there is an $r > 0$ and $\mathbb{R}^{2n} \cong \mathbb{R}^2 \oplus \mathbb{R}^{2n}$ s.th.

- $(S_n - x) \cap B_r^2(0) \times B_r^{2n-2}(0) = \operatorname{graph}(f_n)$
- $(S - x) \cap B_r^2(0) \times B_r^{2n-2}(0) = \operatorname{graph}(f)$

• $f_n \to f$ in $C^{k-1,\alpha}$ for every $\alpha \in [0,1)$

Rmk: (S_n) weakly C^k converges to S for all k

$\Longrightarrow$ (S_n) smoothly conv. to S.

③ Prop (Arzelá–Ascoli):

$$r^k_{S_n} \geq \varrho > 0 \Rightarrow \text{a subseq. of } (S_n)$$

weakly C^k conv.

④ Lemma: There are $\varepsilon, \delta > 0$ s.th. if

for some $\varrho > 0$ and every $x \in S$, $s \in (0, \varrho]$:

$$\Theta_S(x,s) = \frac{\mathcal{H}^2(S \cap B_s(x))}{\pi s^2} \leq 1 + \varepsilon \quad \text{and}$$

S is J-hol.,

then

$$r^k_S \geq \delta \cdot \varrho.$$

Pf: if not, .. rescale .. limit .. Schauder .. ↯

↑

"Compactness from rigidity"

must be plane $\mathbb{R}^2$ by monotonicity

⑤ If (δ_{S_n}) weakly conv. to δ_S, then

$\Theta_{S_n}(x,s) \leq 1 + \varepsilon$ for $s \ll 1$ and $n \gg 1$.

Lemma + Prop $\Rightarrow$ (S_n) smoothly conv. to S. ∎