# Differential Geometry IV
# Summer Semester 2026

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This instance of Differential Geometry IV is a course on *spin geometry*.

**Prerequisites**   As prerequisites I will assume a firm understanding of the theory of smooth manifolds, vector bundles, covariant derivatives, etc. and some familiarity with Riemannian geometry. Prior exposure to the theory of $G$–principal bundles might be helpful, but is not necessary. Students who have successfully participated in Differential Geometry I and II at HU Berlin should have no trouble following this course.

**Topics**   Spin geometry is a vast subject and area of active research. The purpose of this course is to lead the students from the beginnings of spin geometry to some parts (of the vicinity of) the frontier of our current knowledge. Of course, not everything can be covered in one semester. However, I somewhat firmly plan to discuss at least the following:

(1)  The framework of spin geometry.

    (a)  Motivations: (i) Dirac's work on $\sqrt{\Delta}$? (ii) Rokhlin's theorem and the Hirzebruch signature theorem (iii) an extremely vague sketch of Seiberg–Witten theory (iv) Positive scalar curvature.

    (b)  Clifford algebras, Dirac bundles, and Dirac operators.

    (c)  The Schrödinger–Lichnerowicz formula.

    (d)  The spin group, spin structures, the Atiyah–Singer operator.

    (e)  Dirac operators on homogeneous spaces $G/H$.

    (f)  Weierstraß representation of minimal surfaces from the spinorial point of view [Fri98].

(2)  Analysis of the Dirac operator.

    (a)  A review of $L^2$ elliptic theory.

    (b)  The spectrum of the Dirac operator: lower bounds, model computations, Weyl's law.

    (c)  The space of Dirac minimal metrics [BG92; Mai97; Ang96; AD25].

    (d)  Dirac operators and surgery [BD02].

(3) Positive scalar curvature.

    (a) A glimpse of the work of Gromov and Lawson [GL83].

    (b) Llarull's Theorem [Lla98] and recent developments around it, e.g., [BZ25]

There are a number of further topics that could be discussed based on the students' interests and the available remaining time. These include but are not limited to:

(1) The index theorem.

(2) Seiberg–Witten theory.

(3) Spin geometry with boundary: boundary value probems, fill-ins, ...

# References

[AD25]    Bernd Ammann and Mattias Dahl. *The space of Dirac-minimal metrics is connected in dimensions 2 and 4*. English. *SIGMA, Symmetry Integrability Geom. Methods Appl.* 21 (2025), paper 102, 18. DOI: 10.3842/SIGMA.2025.102 (cit. on p. 1)

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[BZ25]    Christian Baer and Rudolf Zeidler. *The degree condition in Llarull's theorem on scalar curvature rigidity*. 2025. arXiv: 2507.05459 [math.DG]. (cit. on p. 2)

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[BG92]    J.-P. Bourguignon and P. Gauduchon. *Spineurs, opérateurs de Dirac et variations de métriques*. Communications in Mathematical Physics 144.3 (1992), pp. 581–599. DOI: 10.1007/BF02099184. MR: 1158762. Zbl: 0755.53009 (cit. on p. 1)

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[Mai97]    S. Maier. *Generic metrics and connections on* Spin– *and* Spin$^c$ *–manifolds*. Communications in Mathematical Physics 188.2 (1997), pp. 407–437. DOI: 10.1007/s002200050171. MR: 1471821. Zbl: 0899.53036 (cit. on p. 1)