Differential Geometry IV
Problem Set 5

Prof. Dr. Thomas Walpuski

2022-05-12

Due: 2022-05-25

(1) Let X be a Riemannian spin manifold. Let $Y \subset X$ be an oriented submanifold. Work out how to restrict the spin structure of X to Y! Which problems might arise restricting spin structures in the pseudo-Riemannian case?

(2) Explicitly describe the two spin structures on S^1! Which one is the restriction of the unique spin structure on D^2?

(3) Let (V, g) be a Euclidean vector space. Let (S, γ) be a $\mathrm{C}\ell(g)$–module. Let b be a bilinear form with respect to which $\gamma(v)$ is skew-adjoint for every $v \in V$. Define $\tilde{\cdot}\colon \mathfrak{o}(V) \to \mathfrak{o}(S)$ by

$$\tilde{A} \coloneqq \sum_{i=1}^{n} g(ae_i, e_j)[\gamma(e_i), \gamma(e_j)]$$

for $(e_1, \ldots, e_n)$ an orthonormal basis. Check that this does not depend on the choice of orthonormal basis. Determine the relation between

$$[\tilde{A}, \gamma(v)] \quad \text{and} \quad \gamma(Av).$$

(This underlies the discussion of the refined Weitzenböck formula in the lecture notes.)

(4) Let (X, g) be a connected spin manifold. Denote by (S, γ, b, ∇) the corresponding Dirac bundle. Prove that g is Ricci-flat if there is a non-zero $\phi \in \Gamma(S)$ with $\nabla\phi = 0$. (*Hint:* Determine a formula for the curvature of ∇ based on the discussion of the refined Weitzenböck formula.)

(5) Establish a pullback diagram

$$\begin{array}{ccc} \mathrm{Spin}^{\mathrm{U}(1)}_{r,s} & \longrightarrow & \mathrm{Spin}_{r,s+2} \\ \downarrow & & \downarrow \\ \mathrm{SO}^{+}_{r,s} \times \mathrm{U}(1) & \longrightarrow & \mathrm{SO}^{+}_{r,s+2}. \end{array}$$

Use this to prove that V admits a $\mathrm{spin}^{\mathrm{U}(1)}$ structure if and only if there is a Hermitian line bundle L such that $V \oplus L$ admits a spin structure. Finally, prove the following.

Proposition 0.1.

(a) *V admits a $\mathrm{spin}^{\mathrm{U}(1)}$ structure if and only if $w_2(V) \in \mathrm{im}(\mathrm{H}^2(X, \mathbf{Z}) \to \mathrm{H}^2(X, \mathbf{Z}/2\mathbf{Z})$ if and only if $W_3(V) = 0$.*

(b) *If V admits a $\mathrm{spin}^{\mathrm{U}(1)}$ structure, then the set of $\mathrm{spin}^{\mathrm{U}(1)}$ structures is a torsor over $\mathrm{H}^2(X, \mathbf{Z})$.*

(6) Suppose that V admits a spin structure. Describe the set of all spin structure on V inducing the same $\mathrm{spin}^{\mathrm{U}(1)}$ structure. Describe the of all $\mathrm{spin}^{\mathrm{U}(1)}$ structures on V with trivial characteristic line bundle.