# Differential Geometry IV
# Problem Set 4

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**Due**: 2022-05-18

(1) Let $D \in \{\mathbf{R}, \mathbf{C}, \mathbf{H}\}$ Let $P$ be a finite-dimensional $D \otimes \mathrm{C}\ell_{0,n}$–module. Prove that there is a Euclidean inner product $\langle\cdot,\cdot\rangle$ on $P$ such that

$$\langle v\phi, v\psi\rangle = |v|^2\langle\phi,\psi\rangle$$

and, moreover, $i$ is orthogonal if $D = \mathbf{C}$ and $i$, $j$, $k$ are orthogonal if $D = \mathbf{H}$.

(2) Establish the exceptional isomorphisms

(a) $\mathrm{Spin}_{0,3} \cong \mathrm{Sp}(1) \cong \mathrm{SU}(2)$.
(b) $\mathrm{Spin}_{1,2} \cong \mathrm{SL}_2(\mathbf{R})$.
(c) $\mathrm{Spin}_{0,4} \cong \mathrm{Sp}(1)\times\mathrm{Sp}(1) \cong \mathrm{SU}(2)\times\mathrm{SU}(2)$.
(d) $\mathrm{Spin}_{1,3} \cong \mathrm{SL}_2(\mathbf{C})$.
(e) $\mathrm{Spin}_{0,5} \cong \mathrm{Sp}(2)$.
(f) $\mathrm{Spin}_{0,6} \cong \mathrm{SU}(4)$.

(3) Since $\mathrm{Spin}_{r,s} \subset (\mathrm{C}\ell^0_{r,s})^\times$ is a Lie subgroup, $\mathfrak{spin}_{r,s} \subset \mathrm{C}\ell^0_{r,s}$ with the Lie bracket agreeing with the commutator. Set $b = b_{r,s} \coloneqq \frac{1}{2}p_{r,s}$, $\frac{1}{2}$ of the polarisaton of $q_{r,s}$. Denote by $\kappa\colon \Lambda^2\mathbf{R}^{r+s} \to \mathrm{C}\ell^0_{r,s}$ the map induced by the quantisation map. Identify $\Lambda^2\mathbf{R}^{r+s} = \mathfrak{so}_{r,s}$ via

$$(u\wedge v)w \coloneqq ub_{r,s}(v,x) - vb_{r,s}(u,x).$$

Prove the following:

(a) $\mathfrak{spin}_{r,s}$ agrees with the image of the quantisation map $\kappa\colon \Lambda^2\mathbf{R}^{r+s} \to \mathrm{C}\ell^0_{r,s}$.
(b) $\mathrm{Lie}(\mathrm{Ad})\circ\kappa(\alpha) = 2\alpha$ for every $\alpha \in \Lambda^2\mathbf{R}^{r+s} = \mathfrak{so}_{r,s}$.

(4) What double covers $\mathrm{Pin}^{\star}_{r,s} \to \mathrm{O}_{r,s}$ are there such that the following diagram commutes

$$\begin{array}{ccc}
\{\pm 1\} & = & \{\pm 1\} \\
\downarrow & & \downarrow \\
\mathrm{Spin}^{+}_{r,s} & \longrightarrow & \mathrm{Pin}^{\star}_{r,s} \\
\downarrow & & \downarrow \\
\mathrm{SO}^{+}_{r,s} & \longrightarrow & \mathrm{O}_{r,s}?
\end{array}$$