

The universal moduli space of non-degenerate $\mathbb{Z}/2\mathbb{Z}$ harmonic spinors

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Abstract

This article equips the universal moduli space of non-degenerate $\mathbb{Z}/2\mathbb{Z}$ harmonic spinors with the structure of a tame Fréchet manifold and proves that the projection map to the parameter space is uniformly Fredholm. This recovers earlier work of Donaldson [Don21a], Parker [Par23], and Takahashi [Tak23] in a unified framework.

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1 Introduction

More than a decade ago, Taubes and his disciples began to observe that the failure of compactness for a wide variety of “new” low-dimensional gauge theories leaves behind evidence in the form of a $\mathbb{Z}/2\mathbb{Z}$ **harmonic spinor** [Tau13a; Tau13b; Tau16; Tau17; Tau22; HW15; WZ21]. The latter have since appeared in related questions in gauge theory and calibrated geometry [Tau14; Don17; DW21; He23], and might play a role in the conjectural construction of invariants of manifolds with special holonomy [DS11; Hay17; Joy18; DW19; DW24].

In light of the above it seems pressing to understand what kind of a phenomenon the existence of $\mathbb{Z}/2\mathbb{Z}$ harmonic spinors is; in particular, whether it persists under (small) deformations of the background parameters. This question of the deformation theory of $\mathbb{Z}/2\mathbb{Z}$ harmonic spinors has been approached before in the work of Donaldson [Don21a], Parker [Par23], and Takahashi [Tak23]. The purpose of this article is to develop it in a unified framework building upon [BW26] and a suggestion by Donaldson [Don21b].

Throughout the remainder of this article, let X be a connected closed manifold of dimension d , and S a Euclidean vector bundle of rank r over X . Denote by $\mathbf{Met}$ the tame Fréchet manifold of Riemannian metrics g on X , by $\mathbf{Dir}$ the tame Fréchet manifold of Dirac bundle structures $\mathbf{p} = (g; \gamma, \nabla)$ on S (as in [LM89, Part II Definition 5.2] and Section 2), and by $\mathfrak{Ram}$ the set of isometry classes of ramified Euclidean line bundles $[I]$ over X . The goal is to equip the universal set of (non-zero) $\mathbb{Z}/2\mathbb{Z}$ harmonic spinors (up to scaling)

$$\mathbf{PHarm} := \bigsqcup_{(\mathbf{p}, [I]) \in \mathbf{Dir} \times \mathfrak{Ram}} \mathbf{P} \ker(D_{\mathbf{p}}^I : H^1 \Gamma(X \setminus \text{Br}(I), S \otimes I) \rightarrow L^2 \Gamma(X \setminus \text{Br}(I), S \otimes I))$$

with some geometric structure and understand the nature of the projection map $\text{pr}_{\mathbf{Dir}} : \mathbf{PHarm} \rightarrow \mathbf{Dir}$. Here $D_{\mathbf{p}}^I$ denotes the Dirac operator associated with $\mathbf{p}$ twisted by I , and $\mathbf{P}$ denotes the real projectivisation; that is: $\mathbf{P}V := (V \setminus \{0\})/\mathbb{R}^\times$. This structure would upgrade the set $\mathbf{PHarm}$ to the **universal moduli space of $\mathbb{Z}/2\mathbb{Z}$ harmonic spinors**.

The present article does not deal with $\mathbf{PHarm}$ itself, but only with the subset

$$\mathbf{PHarm}^* \subseteq \mathbf{PHarm}$$

consisting of $\mathbb{Z}/2\mathbb{Z}$ harmonic spinors $(\mathbf{p}, [I]; [\Phi])$ which are **non-degenerate** in the following sense:

- (1) the branching locus $\text{Br}(\mathbf{I}) \subseteq X$ is a submanifold, and
- (2) $|\Phi|$ vanishes exactly to order $\frac{1}{2}$ at every $x \in \text{Br}(\mathbf{I})$; that is: $|\Phi| \asymp_\Phi r^{1/2}$. Here r denotes the distance to $\text{Br}(\mathbf{I})$.

The first restriction is imposed because $\mathfrak{Ram}$ is rather wild and it is far from obvious what structure to put on it, unless $d = 2$, see [TW21, §3.1]. The subset $\mathbf{Ram} \subseteq \mathfrak{Ram}$ of isometry classes of ramified Euclidean line bundles $[\mathbf{I}]$ with smooth branching locus $\text{Br}(\mathbf{I})$, however, can be equipped with the structure of a tame Fréchet manifold in a straightforward manner. With this in mind, it is natural to try to equip $\mathbf{PHarm}^*$ with the structure of a tame Fréchet manifold. As explained above, the crucial difficulty is that $D_p^{\mathbf{I}}: H^1\Gamma(X \setminus \text{Br}(\mathbf{I}), S \otimes \mathbf{I}) \rightarrow L^2\Gamma(X \setminus \text{Br}(\mathbf{I}), S \otimes \mathbf{I})$ is left semi-Fredholm but has an ∞ -dimensional cokernel, and the naive expectation is that this can be compensated by allowing $\text{Br}(\mathbf{I})$ to vary. Indeed, provided $|\Phi|$ vanishes exactly to order $\frac{1}{2}$ along $\text{Br}(\mathbf{I})$, this additional variation essentially corresponds to passing to a closed extension of $D_p^{\mathbf{I}}$. If $\text{rk } S = 4$, then this extension is Fredholm. This observation and a suitable framework enable Nash–Moser theory to prove the following.

Theorem 1.1. *If $\dim X = 3$ and $\text{rk } S = 4$, then $\mathbf{PHarm}^*$ can be given the structure of a tame Fréchet manifold such that $\text{pr}_{\mathbf{Dir}}: \mathbf{PHarm}^* \rightarrow \mathbf{Dir}$ is uniformly Fredholm of index -1 .*

The notion of a uniform Fredholm map between tame Fréchet manifolds is defined in Definition 4.1. Crucially, by Proposition 4.3, it is stable under transverse base change. In particular, if B is a (finite-dimensional) manifold and $\Gamma: B \rightarrow \mathbf{Dir}$ is a smooth map which is transverse to $\text{pr}_{\mathbf{Dir}}$, then the pullback $\Gamma^*\mathbf{PHarm}^*$ is a manifold of dimension $\dim B - 1$. In this sense, the appearance of $\mathbb{Z}/2\mathbb{Z}$ harmonic spinors is a codimension one phenomenon in dimension three.

Remark 1.2. If $\dim X = 2$, then $\mathbf{Ram}$ is finite-dimensional, which drastically simplifies the issue. The analogue of Theorem 1.1 in this setting is essentially [DW24, Theorem 3.40]. ♣

Remark 1.3. Theorem 1.1 is essentially [Par23, Theorem 1.5]; see also [Tak23, Theorem 1.5]. The proof of Theorem 1.1 presented in Section 4.3 is based on a suggestion by Donaldson [Don21b] and rather simpler than the argument given in [Par23]. ♣

Remark 1.4. The restriction to $\mathbf{PHarm}^*$ is not unproblematic, since it prevents $\text{pr}_{\mathbf{Dir}}$ from being proper; cf. [TW21, §1.4]. Therefore, it would be interesting to analyse what happens if the non-degeneracy condition is relaxed either by allowing $\text{Br}(\mathbf{I})$ to be singular or $|\Phi|$ to vanish to a higher order at certain points in $\text{Br}(\mathbf{I})$. [HMT23] contains some initial work in the direction of allowing $\text{Br}(\mathbf{I})$ to be a graph. ♣

Remark 1.5. The tame Fréchet manifold structure on $\mathbf{PHarm}^*$ is obtained by exhibiting it as a tame Fréchet submanifold of the projectivisation $\mathbf{PE}_{00}$ of a tame Fréchet space bundle $\mathbf{E}_{00}$ over $\mathbf{Dir} \times \mathbf{Ram}$. Moreover, locally, $\mathbf{PHarm}^*$ can be thickened to a submanifold $\mathbf{PHarm}^{\dagger} \subseteq \mathbf{PE}_{00}$ such that $\mathcal{V} := \text{pr}_{\mathbf{Dir}}(\mathbf{PHarm}^{\dagger})$ is open and $\text{pr}_{\mathcal{V}}: \mathbf{PHarm}^{\dagger} \rightarrow \mathcal{V}$ is a uniform submersion. This additional information can be useful to determine pullbacks of $\mathbf{PHarm}^*$ along tame smooth maps that fail to be transverse to $\text{pr}_{\mathbf{Dir}}$. ♣

The argument in the proof of Theorem 1.1 is somewhat robust to the addition of mild

lower-order terms. As an application one obtains the following result regarding

$$\mathbf{PEig}^* := \coprod_{(\mathbf{p}, [\mathbf{I}], \lambda) \in \mathbf{Dir} \times \mathbf{Ram} \times \mathbb{R}} \mathbf{P}\{\Phi \in H^1\Gamma(X \setminus \text{Br}(\mathbf{I}), S \otimes \mathbf{I}) : \Phi \text{ satisfies (2) and } D_{\mathbf{p}}^{\mathbf{I}}\Phi = \lambda\Phi\},$$

the universal moduli space of non-degenerate $\mathbb{Z}/2\mathbb{Z}$ eigenspinors.

Theorem 1.6 (cf. [Par23, Corollary 1.6]). *If $\dim X = 3$ and $\text{rk } S = 4$, then $\mathbf{PEig}^*$ can be given the structure of a tame Fréchet manifold such that $\text{pr}_{\mathbf{Dir}} : \mathbf{PEig}^* \rightarrow \mathbf{Dir}$ is uniformly Fredholm of index 0 and the projection map $\lambda : \mathbf{PEig}^* \rightarrow \mathbb{R}$ is smooth.*

Of course, $\mathbf{PHarm}^* = \lambda^{-1}(0) \subseteq \mathbf{PEig}^*$.

The hypothesis $\text{rk } S = 4$ is essential in the above. As is explained at the end of Section 3.7, the infinitesimal deformation theory of non-degenerate $\mathbb{Z}/2\mathbb{Z}$ harmonic spinors at $(\mathbf{p}, [\mathbf{I}]; \Phi)$ is governed by the Dirac operator $D_{\mathbf{p}}^{\mathbf{I}}$ subject to a residue condition arising from the inclusion $NZ \hookrightarrow \check{S}$ of the normal bundle of Z into the bundle of possible residue values, encoding the leading order behaviour of Φ at Z . For this residue condition to be elliptic it is necessary that $2 = \text{rk } NZ = \frac{1}{2} \text{rk } \check{S} = \frac{1}{2} \text{rk } S$.

If $\dim X \geq 4$, then $\text{rk } S \geq 8$ for every Dirac bundle S ; however, if X is oriented and $\dim X = 0 \pmod{4}$, then every choice of $\mathbf{p} \in \mathbf{Dir}$ orthogonally decomposes S into

$$S = S^+ \oplus S^- \quad \text{with} \quad S^{\pm} := \ker(\varepsilon \mp 1) \quad \text{and} \quad \varepsilon := \gamma(\text{vol}_g);$$

moreover, for every $[\mathbf{I}] \in \mathbf{Ram}$, $D_{\mathbf{p}}^{\mathbf{I}}$ decomposes accordingly as

$$D_{\mathbf{p}}^{\mathbf{I}} = \begin{pmatrix} 0 & D_{\mathbf{p}}^{\mathbf{I}, -} \\ D_{\mathbf{p}}^{\mathbf{I}, +} & 0 \end{pmatrix}.$$

The universal set of (non-zero) chiral $\mathbb{Z}/2\mathbb{Z}$ harmonic spinors (up to scaling)

$$\mathbf{PHarm}^{\pm} := \coprod_{(\mathbf{p}, [\mathbf{I}]) \in \mathbf{Dir} \times \mathbf{Ram}} \mathbf{P} \ker(D_{\mathbf{p}}^{\mathbf{I}, \pm} : H^1\Gamma(X \setminus \text{Br}(\mathbf{I}), S^{\pm} \otimes \mathbf{I}) \rightarrow L^2\Gamma(X \setminus \text{Br}(\mathbf{I}), S^{\mp} \otimes \mathbf{I}))$$

contains the subset

$$\mathbf{PHarm}^{\pm, *} \subseteq \mathbf{PHarm}^{\pm}$$

of non-degenerate chiral $\mathbb{Z}/2\mathbb{Z}$ harmonic spinors (up to scaling). The restriction of the closed extension of $D_{\mathbf{p}}^{\mathbf{I}}$ considered above leads to a closed extension of $D_{\mathbf{p}}^{\mathbf{I}, \pm}$ which is Fredholm if $\text{rk } S = 8$. The proof of Theorem 1.1 adapts with minor cosmetic modifications to establish the following result.

Theorem 1.7. *If X is oriented, $\dim X = 4$, and $\text{rk } S = 8$, then $\mathbf{PHarm}^{\pm, *}$ can be given the structure of a tame Fréchet manifold such that $\text{pr}_{\mathbf{Dir}} : \mathbf{PHarm}^{\pm, *} \rightarrow \mathbf{Dir}$ is uniformly Fredholm.*

Remark 1.8. If X is oriented, $\dim X = 4$, and $\text{rk } S = 8$, then $\varepsilon := \gamma(\text{vol}_g)$ is the unique chirality operator on S up to sign. Indeed, if δ is another chirality operator, then it commutes with ε ; therefore, $\delta\varepsilon = \varepsilon\delta$ is Clifford linear, self-adjoint, and parallel; hence, by Schur's Lemma $\delta = \pm\varepsilon$. In particular, there is no additional loss of generality in the above setup. ♣

Remark 1.9. The index of pr_{Dir} in [Theorem 1.7](#) is given by the index of the relevant closed extension $D_{\text{p}, R_{\Phi}^{\pm}}^{\text{l}, \pm}$ of $D_{\text{p}}^{\text{l}, \pm}$ minus one. In view of [\[Yano7\]](#), the latter is expected to be given by the index of $D_{\text{p}}^{\pm}$ plus a contribution $\delta(S, \text{I}, \Phi)$ that localises on $\text{Br}(\text{I})$. A brief computation using the Atiyah–Singer index theorem [\[AS68, Theorem 5.3\]](#) gives

$$\text{index } D_{\text{p}, R_{\Phi}^{\pm}}^{\text{l}, \pm} = -\frac{1}{2}p_1(S^{\pm}) + \sigma(X) \mp \frac{1}{2}e(TX) + \delta(S, \text{I}, \Phi) = -\frac{1}{4}p_1(S) + \sigma(X) + \delta(S, \text{I}, \Phi).$$

[Remark 4.35](#) explains this index computation and how to determine $\delta(S, \text{I}, \Phi)$ in a special case. ♣

The above results (and their proof) can be brought to bear on $\mathbb{Z}/2\mathbb{Z}$ harmonic 1-forms as follows. Assume that $\dim X = 3$ and $S = \mathbb{R} \oplus T^*X$. For every $g \in \mathbf{Met}$ there is a Dirac bundle structure on S whose associated Dirac operator is

$$D_g := \begin{pmatrix} 0 & d^* \\ d & *d \end{pmatrix} : \Omega^0(X) \oplus \Omega^1(X) \rightarrow \Omega^0(X) \oplus \Omega^1(X),$$

a truncation of the **Hodge–de Rham operator** $d + d^* : \Omega^{\bullet}(X) \rightarrow \Omega^{\bullet}(X)$. This defines a tame smooth map $\iota_{\text{HdR}} : \mathbf{Met} \rightarrow \mathbf{Dir}$. By integration by parts, if $(f, \alpha) \in \Omega^0(X, \text{I}) \oplus \Omega^1(X, \text{I})$ is $\mathbb{Z}/2\mathbb{Z}$ harmonic, then $f = 0$ —except in the edge case $\text{Br}(\text{I}) = \emptyset$, which shall be disregarded in the remaining discussion. Therefore, the pullback

$$\mathbf{PHarm}_{\Omega^1}^* := \iota_{\text{HdR}}^* \mathbf{PHarm}^*$$

is the **universal moduli space of non-degenerate $\mathbb{Z}/2\mathbb{Z}$ harmonic 1-forms**. Although ι_{HdR} is *not transverse* to $\text{pr} : \mathbf{Harm}^* \rightarrow \mathbf{Dir}$, the theory underlying the proof of [Theorem 1.1](#), see [Remark 1.5](#), provides detailed information on $\mathbf{PHarm}_{\Omega^1}^*$ and the projection map $\text{pr}_{\mathbf{Met}} : \mathbf{PHarm}_{\Omega^1}^* \rightarrow \mathbf{Met}$.

There is a finite rank vector bundle $\mathcal{H}^1 \rightarrow \mathbf{Met} \times \mathbf{Ram}$ whose fibre over $(g, [\text{I}])$ is $\mathcal{H}^1(g, \text{I})$, the space of L^2 harmonic 1-forms $\alpha \in L^2\Omega^1(X, \text{I})$. A singular version of Hodge theory [\[Tel83\]](#) induces an isomorphism $\mathcal{H}^1(g, \text{I}) \cong H_{\text{dR}}^1(\tilde{X})^-$. Here $\tilde{X} \rightarrow X$ denotes the branched double cover induced by I and the superscript indicates taking the anti-invariant part with respect to the sheet-swapping involution. Evidently, there is a canonical inclusion map $j : \mathbf{PHarm}_{\Omega^1}^* \hookrightarrow \mathbf{P}\mathcal{H}^1$.

Theorem 1.10. *If $\dim X = 3$, then $\mathbf{PHarm}_{\Omega^1}^*$ can be given the structure of a tame Fréchet manifold such that:*

- (1) $\text{pr}_{\mathbf{Met}} : \mathbf{PHarm}_{\Omega^1}^* \rightarrow \mathbf{Met}$ is a uniform submersion, and
- (2) j is smooth and induces an isomorphism $T\mathbf{PHarm}_{\Omega^1}^*/\mathbf{Met} \cong j^*T\mathbf{P}\mathcal{H}^1/(\mathbf{Met} \times \mathbf{Ram})$ of the vertical tangent bundles.

In particular, the index of $\text{pr}_{\mathbf{Met}}$ at $(g, [\text{I}]; [\alpha]) \in \mathbf{PHarm}_{\Omega^1}^$ agrees with $\dim H_{\text{dR}}^1(\tilde{X})^- - 1$.*

Remark 1.11. [Theorem 1.10](#) recovers [\[Don21a, Theorem 1.1\]](#) in dimension three; see also [\[HP24\]](#) for an adaptation of [\[Par23\]](#) to $\mathbb{Z}/2\mathbb{Z}$ harmonic 1-forms. The proof of [Theorem 1.10](#) presented in [Section 4.6](#) is different from the proof in [\[Don21a\]](#), but, of course, related to the alternative proof of [\[Don21a, Theorem 1.1\]](#) mentioned in [\[Don21b\]](#). ♣

Remark 1.12. The vector bundle $\mathcal{H}^1$ has a flat Gauß–Manin connection. In particular, if $\mathcal{U} \subseteq \mathbf{Met} \times \mathbf{Ram}$ is a simply-connected neighbourhood of $(g, [\mathbf{I}])$, then it induces a trivialisation $\mathcal{H}^1|_{\mathcal{U}} \cong H_{\mathrm{dR}}^1(\tilde{X})^-$. As a consequence of [Theorem 1.10](#), every $(g, [\mathbf{I}]; [\alpha]) \in \mathbf{PHarm}_{\Omega^1}^*$ has an open neighbourhood $\mathcal{U} \subseteq \mathbf{PHarm}_{\Omega^1}^*$ such that there is a well-defined **local period map** $\mathbf{Per}: \mathcal{U} \rightarrow \mathbf{PH}_{\mathrm{dR}}^1(\tilde{X})^-$ and $\mathrm{pr}_{\mathbf{Met}} \times \mathbf{Per}: \mathcal{U} \rightarrow \mathbf{Met} \times \mathbf{PH}_{\mathrm{dR}}^1(\tilde{X})^-$ is an open embedding. This formulation is closer to the one found in [\[Don21a, Theorem 1.1\]](#). ♣

There is a variant of the preceding discussion assuming that $\dim X = 4$, X is oriented, and $S = \mathbb{R} \oplus \Lambda^+ T^*X \oplus T^*X$. For every $g \in \mathbf{Met}$ there is a Dirac bundle structure on S whose associated Dirac operator is

$$D_g := \begin{pmatrix} 0 & d & \sqrt{2}d^* \\ d^* & 0 & 0 \\ \sqrt{2}d^+ & 0 & 0 \end{pmatrix}: \Omega^1(X) \oplus \Omega^0(X) \oplus \Omega^+(X) \rightarrow \Omega^1(X) \oplus \Omega^0(X) \oplus \Omega^+(X).$$

This defines a tame smooth map $\iota_{\mathrm{HdR}}: \mathbf{Met} \rightarrow \mathbf{Dir}$. A brief computation shows that

$$\mathbf{PHarm}_{\Omega^1}^* := \iota_{\mathrm{HdR}}^* \mathbf{PHarm}^{+,*} \quad \text{and} \quad \mathbf{PHarm}_{\Omega^+}^* := \iota_{\mathrm{HdR}}^* \mathbf{PHarm}^{-,*}$$

are the **universal moduli spaces of non-degenerate $\mathbb{Z}/2\mathbb{Z}$ harmonic 1-forms and self-dual 2-forms** respectively. There are finite rank vector bundles $\mathcal{H}^1 \rightarrow \mathbf{Met} \times \mathbf{Ram}$ and $\mathcal{H}^+ \rightarrow \mathbf{Met} \times \mathbf{Ram}$ whose fibres over $(g, [\mathbf{I}])$ are $\mathcal{H}^1(g, \mathbf{I})$, as above, and $\mathcal{H}^+(g, \mathbf{I})$, the space of L^2 harmonic self-dual 2-forms $\alpha \in L^2 \Omega^+(X, \mathbf{I})$. As before, $\mathcal{H}^1(g, \mathbf{I}) \cong H_{\mathrm{dR}}^1(\tilde{X})^-$; moreover, $\mathcal{H}^+(g, \mathbf{I})$ is isomorphic to $H_{\mathrm{dR}}^+(\tilde{X})^-$, a maximal subspace of $H_{\mathrm{dR}}^2(\tilde{X})^-$ on which the intersection form is positive definite. There are canonical inclusions $j: \mathbf{PHarm}_{\Omega^1}^* \hookrightarrow \mathbf{P}\mathcal{H}^1$ and $j: \mathbf{PHarm}_{\Omega^+}^* \hookrightarrow \mathbf{P}\mathcal{H}^+$.

Theorem 1.13. *If $\dim X = 4$, then:*

(1) $\mathbf{PHarm}_{\Omega^1}^*$ can be given the structure of a tame Fréchet manifold such that:

- (a) $\mathrm{pr}_{\mathbf{Met}}: \mathbf{PHarm}_{\Omega^1}^* \rightarrow \mathbf{Met}$ is a uniform submersion, and
- (b) j is smooth and induces an isomorphism $T\mathbf{PHarm}_{\Omega^1}^*/\mathbf{Met} \cong j^*T\mathbf{P}\mathcal{H}^1/(\mathbf{Met} \times \mathbf{Ram})$.

In particular, the index of $\mathrm{pr}_{\mathbf{Met}}$ at $(g, [\mathbf{I}]; [\alpha])$ agrees with $\dim H_{\mathrm{dR}}^1(\tilde{X})^- - 1$.

(2) $\mathbf{PHarm}_{\Omega^+}^*$ can be given the structure of a tame Fréchet manifold such that:

- (a) $\mathrm{pr}_{\mathbf{Met}}: \mathbf{PHarm}_{\Omega^+}^* \rightarrow \mathbf{Met}$ is a uniform submersion, and
- (b) j is smooth and induces an isomorphism $T\mathbf{PHarm}_{\Omega^+}^*/\mathbf{Met} \cong j^*T\mathbf{P}\mathcal{H}^+/(\mathbf{Met} \times \mathbf{Ram})$.

In particular, the index of $\mathrm{pr}_{\mathbf{Met}}$ at $(g, [\mathbf{I}]; [\alpha])$ agrees with $\dim H_{\mathrm{dR}}^+(\tilde{X})^- - 1$.

Remark 1.14. The assertion about $\mathbf{PHarm}_{\Omega^1}^*$ is again nothing but [\[Don21a, Theorem 1.1\]](#) in dimension four. However, the assertion about $\mathbf{PHarm}_{\Omega^+}^*$ is novel. In fact, for self-dual 2-forms the reduction to a scalar equation used in [\[Don21a\]](#) is not available. ♣

Preliminaries This article is based on the theory of tame Fréchet manifolds as developed by Hamilton [Ham82]. Readers are assumed either to be familiar with this subject or to be willing to accept it as a black box. It should be pointed out that, as a small deviation from [Ham82], instead of [Ham82, Part II Definition 1.3.2] the following definition is in effect.

Definition 1.15. A graded Fréchet space $(F, (\|\cdot\|_k)_{k \in \mathbb{N}_0})$ is **tame** if it has the **smoothing operator property**; that is: there is a family of smoothing operators $(S_\varepsilon : \varepsilon \in (0, 1))$ on F such that:

(S1) For every $\varepsilon \in (0, 1)$, $k, \ell \in \mathbb{N}_0$ with $k \geq \ell$, and $x \in F$

$$\|S_\varepsilon x\|_k \lesssim \varepsilon^{\ell-k} \|x\|_\ell.$$

(S2) For every $\varepsilon \in (0, 1)$, $k, \ell \in \mathbb{N}_0$ with $k \leq \ell$, and $x \in F$

$$\|(S_\varepsilon - 1)x\|_k \lesssim \varepsilon^{\ell-k} \|x\|_\ell.$$

(S3) For every $\varepsilon \in (0, 1)$, $k, \ell \in \mathbb{N}_0$, and $x \in F$

$$\varepsilon \|\partial_\varepsilon S_\varepsilon x\|_k \lesssim \varepsilon^{\ell-k} \|x\|_\ell. \quad \bullet$$

This is a weaker condition than [Ham82, Part II Definition 1.3.2], but sufficient for the proof of the Nash–Moser inverse function theorem [Ham82, Part III Theorem 1.1.1].

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2 Dirac operators twisted by ramified line bundles

The technical foundation of this article is [BW26]. The latter is, of course, inspired by [BB12] and most of its results could plausibly be cobbled together from the powerful machinery developed, e.g., in [Maz91; MV14; AG16; AG23].

For the readers' convenience this section provides an *informal* overview of some of the material in [BW26]. Readers who are familiar with it can skim this section or skip it entirely. Throughout this section, the following data are assumed to be fixed:

- (1) a Riemannian metric g on X ;
- (2) a Dirac bundle structure (γ, ∇) on S with respect to g ; that is:

- (a) a linear map $\gamma: TX \rightarrow \mathfrak{o}(S)$, the **Clifford multiplication**, such that for every $v \in TX$

$$\gamma(v)^2 = -|v|^2 \mathbf{1}_S,$$

and

- (b) a covariant derivative $\nabla: \Gamma(X, S) \rightarrow \Omega^1(X, S)$, the **spin connection**, such that for every $v, w \in \text{Vect}(X)$ and $\phi \in \Gamma(X, S)$

$$\nabla_v(\gamma(w)\phi) = \gamma(\nabla_v w)\phi + \gamma(w)\nabla_v \phi;$$

and

- (3) a **ramified Euclidean line bundle** over X ; that is:

- (a) a closed subset $Z =: \text{Br}(\mathbf{I}) \subseteq X$, the **branching locus**, and
- (b) a Euclidean line bundle $\mathbf{I}$ over $X \setminus Z$

such that

- (c) if $W \subseteq Z$ is closed and $\mathbf{I}$ extends over $X \setminus W$, then $W = Z$.

Moreover, it is *assumed* that Z is a submanifold, necessarily of codimension two.

Associated with these data is a **Dirac operator**

$$D: \Gamma(X \setminus Z, S \otimes \mathbf{I}) \rightarrow \Gamma(X \setminus Z, S \otimes \mathbf{I})$$

defined by

$$D\phi := \sum_{i=1}^d \gamma(e_i) \nabla_{e_i} \phi.$$

Here $(e_1, \dots, e_d)$ is an arbitrary local orthonormal frame and ∇ is the twist of the spin connection by the unique orthogonal connection on $\mathbf{I}$.

2.1 The Gelfand–Robbin quotient

To begin putting the Dirac operator into a functional analytic context, consider the **minimal extension**

$$D_{\min} = D: \text{dom}(D_{\min}) := H^1 \Gamma(X \setminus Z, S \otimes \mathbf{I}) \rightarrow L^2 \Gamma(X \setminus Z, S \otimes \mathbf{I}).$$

$D_{\min}$ is left semi-Fredholm but its cokernel is ∞ -dimensional [BW26, Proposition 2.3 and Lemma 3.3]. Its adjoint, in the sense of unbounded operators, is the **maximal extension**

$$D_{\max} = D_{\min}^*: \text{dom}(D_{\max}) \rightarrow L^2 \Gamma(X \setminus Z, S \otimes \mathbf{I});$$

in fact, by [BW26, Remark 2.10],

$$\text{dom}(D_{\max}) = \{\phi \in H_{\text{loc}}^1 \Gamma(X \setminus Z, S \otimes \mathbf{I}) : \phi, D\phi \in L^2 \Gamma(X \setminus Z, S \otimes \mathbf{I})\}.$$

Of course, $D_{\max}$ is right semi-Fredholm but its kernel is ∞ -dimensional.

The intermediate extensions of $D_{\min}$ are controlled by the **Gelfand–Robbin quotient**

$$\check{\mathbf{H}} := \frac{\text{dom}(D_{\max})}{\text{dom}(D_{\min})}.$$

This is a symplectic Hilbert space; its symplectic form, the **Green’s form**, is defined by

$$G([\phi], [\psi]) := \langle D\phi, \psi \rangle_{L^2} - \langle \phi, D\psi \rangle_{L^2}.$$

Every closed extension of $D_{\min}$ is of the form

$$D_R := D_{\max}|_{\text{dom}(D_R)} \quad \text{with} \quad \text{dom}(D_R) = \{\phi \in \text{dom}(D_{\max}) : [\phi] \in R\};$$

where $R \subseteq \check{\mathbf{H}}$ is a closed subspace [BW26, Proposition 2.14]; moreover,

$$D_R^* = D_{R^G};$$

that is: the adjoint of D_R is obtained by forming the symplectic complement R^G of R [BW26, Proposition 2.17]. As in [BW26, Definition 2.13], a **residue condition** is a closed subspace $R \subseteq \check{\mathbf{H}}$.

2.2 The residue map

In an abstract way, $[\phi] \in \check{\mathbf{H}}$ encodes the leading-order behaviour of $\phi \in \text{dom}(D_{\max})$ near Z . This leads to the following more geometric realisation of the Gelfand–Robbin quotient.

It is instructive to begin by considering an untwisted spinor $\phi \in \Gamma(X, S)$ vanishing along Z . The Taylor expansion of ϕ at Z has a leading-order term $\phi^{(1)} \in \Gamma(Z, \text{Hom}(NZ, S|_Z))$. A brief computation shows that if $D\phi = 0$, then $\phi^{(1)}$ satisfies the algebraic constraint

$$\phi^{(1)} \circ \text{or} = -\gamma(\text{or}) \circ \phi^{(1)}$$

for every $x \in Z$ and $\text{or} \in \Lambda^2 N_x Z$. Here $\Lambda^2 N_x Z$ is identified with $\mathfrak{o}(N_x Z)$ using the Riemannian metric g so that $(u \wedge v)(w) = \langle u, w \rangle v - \langle v, w \rangle u$ as in [LM89, Part I §6 (6.3)].

If Z is cooriented, then there is a preferred $\text{or} \in \Lambda^2 NZ$ of unit length which defines complex structures on $S|_Z$ and NZ . The algebraic constraint is then simply that $\phi^{(1)}$ is complex anti-linear. However, Z might not be coorientable. Nevertheless, the Riemannian metric defines an isomorphism $(\Lambda^2 NZ)^{\otimes 2} \cong \mathbb{R}$ and this produces a flat bundle

$$\mathbb{A} = \mathbb{R} \oplus i\Lambda^2 NZ$$

of normed $\mathbb{R}$ -algebras over Z whose fibres are isomorphic to $\mathbb{C}$, but canonically only up to complex conjugation. The preceding discussion exhibits both NZ and $S|_Z$ as $\mathbb{A}$ -modules. If $D\phi = 0$, then $\phi^{(1)}$ is $\mathbb{A}$ -anti-linear.

The analogous construction for $\phi \in \text{dom}(D_{\max})$ is quite a bit more involved. Its outcome is very briefly stated as follows. The Dirac bundle structure on S restricts to a Dirac bundle structure on $S|_Z$ [BW26, Definition 3.9]. As explained in [BW26, Definition 3.21 and Remark

3.22], the Euclidean line bundle $\mathbf{I}$ defines an $\mathbb{A}$ -module $NZ^{-1/2}$ together with an $\mathbb{A}$ -linear orthogonal connection, and an isomorphism

$$NZ^{-1/2} \otimes_{\mathbb{A}} NZ^{-1/2} \cong \text{Hom}_{\mathbb{A}}(NZ, \mathbb{A}).$$

This endows the **residue bundle**

$$\check{S} := S|_Z \otimes_{\mathbb{A}} NZ^{-1/2}$$

with a Dirac bundle structure; in particular, $\check{S}$ has a Dirac operator $D_{\check{S}}$. Moreover, the restriction of the Clifford multiplication to NZ induces a $\mathbb{A}$ -anti-linear almost complex structure

$$J \in \Gamma(Z, \overline{\text{End}_{\mathbb{A}}(\check{S})}).$$

This together with the Euclidean inner product induces a symplectic structure $\check{\Omega}$ on $\check{S}$. Roughly speaking, after identifying $N_x Z = \mathbb{C}$, an element of $S|_Z \otimes_{\mathbb{A}} NZ^{-1/2}$ can be viewed as a term of the form $\phi \bar{z}^{-1/2}$. This should make it plausible that the Gelfand–Robbin quotient $\check{\mathbf{H}}$ can be realised as a suitable completion of $\Gamma(Z, \check{S})$.

The norm with respect to which $\Gamma(Z, \check{S})$ has to be completed is defined in terms of the **branching locus operator**

$$A := -JD_{\check{S}}.$$

Since J is skew-adjoint, $D_{\check{S}}$ is (formally) self-adjoint, and J and $D_{\check{S}}$ anti-commute, A is (formally) self-adjoint. Denote by $\mathbf{1}_{(-\infty, 0)}(A)$ and $\mathbf{1}_{[0, \infty)}(A)$ the orthogonal projections to the negative and non-negative eigenspaces of A respectively. Define the norm $\|\cdot\|_{\check{H}}: \Gamma(Z, \check{S}) \rightarrow [0, \infty)$ by

$$\|\phi\|_{\check{H}} := \|\mathbf{1}_{(-\infty, 0)}(A)\phi\|_{H^{1/2}} + \|\mathbf{1}_{[0, \infty)}(A)\phi\|_{H^{-1/2}},$$

and denote by $\check{H}\Gamma(Z, \check{S})$ the completion of $\Gamma(Z, \check{S})$ with respect to $\|\cdot\|_{\check{H}}$. The symplectic form $\check{\Omega}$ on $\check{S}$ induces a symplectic form on $\check{H}\Gamma(Z, \check{S})$ which shall also be denoted by $\check{\Omega}$.

[BW26, Definition 3.40 (1)] constructs the **residue map**

$$\text{res}: \text{dom}(D_{\max}) \rightarrow \check{H}\Gamma(Z, \check{S})$$

which satisfies

$$\ker \text{res} = \text{dom}(D_{\min})$$

and is surjective; in fact, [BW26, Definition 3.40 (2)] constructs a right inverse, the **extension map**

$$\text{ext}: \check{H}\Gamma(Z, \check{S}) \rightarrow \text{dom}(D_{\max}).$$

As a consequence of this, res induces an isomorphism

$$\check{\mathbf{H}} \cong \check{H}\Gamma(Z, \check{S});$$

moreover, through this isomorphism the Green's form corresponds to $\check{\Omega}$. In combination with the discussion in [Section 2.1](#), this shows that every closed extension of $D_{\min}$ corresponds to a residue condition considered as a closed subspace $R \subseteq \check{H}\Gamma(Z, \check{S})$.

Although the residue map is constructed analytically in [BW26, §3], it can be seen to arise from a suitable restriction map as follows. Choose a tubular neighbourhood of Z and use it to construct a blow-up $\beta: \hat{X} \rightarrow X$ of X along Z . The restriction of β to the boundary $\partial\hat{X}$ is identified with the frame bundle

$$\pi: F := \{v \in NZ : |v| = 1\} \rightarrow Z.$$

Set $\hat{S} := \beta^*S$. Since the inclusion $X \setminus Z \subseteq \hat{X}$ is a homotopy equivalence, $\mathbb{I}$ extends to a Euclidean line bundle $\hat{\mathbb{I}}$ over $\hat{X}$. [BW26, Proposition 3.23] constructs an isomorphism

$$\pi^*NZ^{-1/2} \cong \hat{\mathbb{I}}|_{\partial\hat{X}} \otimes \pi^*\mathbb{A}.$$

In particular, this defines an inclusion

$$\pi^*: \Gamma(Z, \check{S}) \hookrightarrow \Gamma(\partial\hat{X}, \hat{S} \otimes \hat{\mathbb{I}})$$

whose image should be understood to correspond to the fibrewise $-\frac{1}{2}$ Fourier modes. If

$$r: X \setminus Z \rightarrow (0, \infty)$$

denotes a **regularised distance** to Z , that is: r is smooth, positive, and agrees with the distance to Z on a neighbourhood of Z , then it turns out that $\text{dom}(D_{\max}) \cap r^{-1/2}\Gamma(\hat{X}, \hat{S} \otimes \hat{\mathbb{I}})$ is dense in $\text{dom}(D_{\max})$ and that for every $r^{-1/2}\phi \in \text{dom}(D_{\max}) \cap r^{-1/2}\Gamma(\hat{X}, \hat{S} \otimes \hat{\mathbb{I}})$

$$\phi|_{\partial\hat{X}} = \pi^* \text{res}(r^{-1/2}\phi).$$

2.3 Conormal and adapted Sobolev spaces

A vector field $v \in \text{Vect}(\hat{X})$ is **conormal** if $v|_{\partial\hat{X}} \in \text{Vect}(\partial\hat{X})$. Denote the subspace of conormal vector fields by $\text{Vect}_b(\hat{X})$. Denote by $\text{DiffOp}^\bullet_b(S \otimes \mathbb{I})$ the $\mathbf{N}_0$ -filtered ring of differential operators acting on $S \otimes \mathbb{I}$. The space $\text{DiffOp}^\bullet_b(S \otimes \mathbb{I}) \subseteq \text{DiffOp}^\bullet(S \otimes \mathbb{I})$ of **conormal** differential operators is the filtered subring generated by $\Gamma(\hat{X}, \text{End}(\hat{S} \otimes \hat{\mathbb{I}}))$ and differential operators of the form ∇_v with $v \in \text{Vect}_b(\hat{X})$.

For every $k \in \mathbf{N}_0$ the **conormal Sobolev space** $H_b^k\Gamma(X \setminus Z, S \otimes \mathbb{I})$ is defined by

$$H_b^k\Gamma(X \setminus Z, S \otimes \mathbb{I}) := \left\{ \phi \in H_{\text{loc}}^k\Gamma(X \setminus Z, S \otimes \mathbb{I}) : \begin{array}{l} P\phi \in L^2\Gamma(X \setminus Z, S \otimes \mathbb{I}) \text{ for} \\ \text{every } P \in \text{DiffOp}_b^k(S \otimes \mathbb{I}) \end{array} \right\}.$$

Choose a finite subset $\mathcal{P}_b^k \subseteq \text{DiffOp}_b^k(S \otimes \mathbb{I})$ which spans $\text{DiffOp}_b^k(S \otimes \mathbb{I})$ over $\Gamma(\hat{X}, \text{End}(\hat{S} \otimes \hat{\mathbb{I}}))$. Define the norm $\|\cdot\|_{H_b^k}: H_b^k\Gamma(X \setminus Z, S \otimes \mathbb{I}) \rightarrow [0, \infty)$ by

$$\|\phi\|_{H_b^k}^2 := \sum_{P \in \mathcal{P}_b^k} \|P\phi\|_{L^2}^2.$$

Of course, these are the norms that appear in Melrose's b -calculus; see, e.g., [Mel91; Mel93]. They should be regarded as higher regularity analogues of $L^2\Gamma(X \setminus Z, S \otimes \mathbb{I})$. The core of the scale

of conormal Sobolev spaces $(H_b^k \Gamma(X \setminus Z, S \otimes \mathbb{I}), (\|-\|_{H_b^k})_{k \in \mathbb{N}_0})$ is a tame Fréchet space [BW26, Proposition 4.32].

For every $k \in \mathbb{N}_0$ the **adapted Sobolev space** $H_a^{k+1} \Gamma(X \setminus Z, S \otimes \mathbb{I})$ is defined by

$$H_a^{k+1} \Gamma(X \setminus Z, S \otimes \mathbb{I}) := \left\{ \phi \in H_{\text{loc}}^{k+1} \Gamma(X \setminus Z, S \otimes \mathbb{I}) : \begin{array}{l} \phi \in H_b^{k+1} \Gamma(X \setminus Z, S \otimes \mathbb{I}) \text{ and} \\ D\phi \in H_b^k \Gamma(X \setminus Z, S \otimes \mathbb{I}) \end{array} \right\}.$$

Define the norm $\|-\|_{H_a^{k+1}} : H_a^{k+1} \Gamma(X \setminus Z, S \otimes \mathbb{I}) \rightarrow [0, \infty)$ by

$$\|\phi\|_{H_a^{k+1}} := \|\phi\|_{H_b^{k+1}} + \|D\phi\|_{H_b^k}.$$

These should be regarded as higher regularity analogues of $\text{dom}(D_{\max})$. Again, the core of the scale of adapted Sobolev spaces $(H_a^{k+1} \Gamma(X \setminus Z, S \otimes \mathbb{I}), (\|-\|_{H_a^{k+1}})_{k \in \mathbb{N}_0})$ is a tame Fréchet space [BW26, Proposition 4.37].

By [BW26, Lemma 4.12 and Lemma 4.13], for every $k \in \mathbb{N}_0$ the residue map $\text{res} : \text{dom}(D_{\max}) \rightarrow \check{H}\Gamma(Z, \check{S})$ restricts to a bounded surjective linear map

$$\text{res} : H_a^{k+1} \Gamma(X \setminus Z, S \otimes \mathbb{I}) \rightarrow H^{k+1/2} \Gamma(Z, \check{S}).$$

The extension map $\text{ext} : \check{H}\Gamma(Z, \check{S}) \rightarrow \text{dom}(D_{\max})$ restricts to a right-inverse

$$\text{ext} : H^{k+1/2} \Gamma(Z, \check{S}) \rightarrow H_a^{k+1} \Gamma(X \setminus Z, S \otimes \mathbb{I}).$$

Moreover, if $\phi \in H_a^{k+1} \Gamma(X \setminus Z, S \otimes \mathbb{I})$ satisfies $\text{res}(\phi) = 0$, then

$$\nabla_v \phi, r^{-1} \phi \in H_b^k \Gamma(X \setminus Z, S \otimes \mathbb{I})$$

for every $v \in \text{Vect}(X)$ [BW26, Proposition 4.27]. Therefore, $\ker \text{res} \subseteq H_a^{k+1} \Gamma(X \setminus Z, S \otimes \mathbb{I})$ should be regarded as a higher regularity analogue of $\text{dom}(D_{\min}) = H^1 \Gamma(X \setminus Z, S \otimes \mathbb{I})$.

2.4 Elliptic residue conditions

For every residue condition $R \subseteq \check{H}\Gamma(Z, \check{S})$ and every $k \in \mathbb{N}_0$ set

$$H_a^{k+1} \Gamma(X \setminus Z, S \otimes \mathbb{I}; R) := H_a^{k+1} \Gamma(X \setminus Z, S \otimes \mathbb{I}) \cap \text{dom}(D_R)$$

and consider the restriction of D_R to

$$D_R^{(k)} : H_a^{k+1} \Gamma(X \setminus Z, S \otimes \mathbb{I}; R) \rightarrow H_b^k \Gamma(X \setminus Z, S \otimes \mathbb{I}).$$

A residue condition $R \subseteq \check{H}\Gamma(Z, \check{S})$ is **∞ -regular** if for every $k \in \mathbb{N}_0$ and every $\phi \in R$

$$\|\phi\|_{H^{k+1/2}} \lesssim_{R,k} \|\mathbf{1}_{(-\infty, 0)}(A)\phi\|_{H^{k+1/2}} + \|\phi\|_{\check{H}}.$$

It is **elliptic** if both R and R^G are ∞ -regular. In this case, $D_R^{(k)}$ and $D_{R^G}^{(k)}$ satisfy elliptic estimates and elliptic regularity [BW26, Theorem 4.17]; in particular, $D_R^{(k)}$ is Fredholm, and $\ker D_R^{(k)}$ and $\text{coker } D_R^{(k)}$ are independent of $k \in \mathbb{N}_0$ [BW26, Theorem 4.17 and Proposition 4.21].

Evidently, the **APS residue condition** defined by

$$R_{\text{APS}} := \mathbf{1}_{(-\infty, 0)}(A)H^{1/2}\Gamma(Z, \check{S}) \subset \check{H}\Gamma(Z, \check{S})$$

is elliptic. For a **local residue condition**, that is: a residue condition of the form

$$R_V := \check{H}\Gamma(Z, V) \subseteq \check{H}\Gamma(Z, \check{S})$$

for a subbundle $V \subseteq \check{S}$, there is the following symbolic criterion for ellipticity [BW26, Theorem 4.40]: R_V is elliptic if for every $v \in TZ \setminus \{0\}$

$$V \oplus J_Y(v)V = \check{S}.$$

If $\text{rk}_{\mathbb{A}} \check{S} = 2$ and $V \subseteq \check{S}$ is an $\mathbb{A}$ -linear subbundle with $\text{rk}_{\mathbb{A}} V = 1$, then R_V satisfies this criterion because $J_Y(v)$ is skew-adjoint and $\mathbb{A}$ -anti-linear; in particular, R_V is elliptic [BW26, Example 4.44].

2.5 Once more, with chirality

Suppose that (S, γ, ∇) is equipped with a **chirality operator**; that is: a self-adjoint parallel isometry $\varepsilon \in \Gamma(X, \text{O}(S))$ such that

$$\gamma\varepsilon + \varepsilon\gamma = 0.$$

This induces an orthogonal decomposition of S into ∇ -parallel subbundles

$$S = S^+ \oplus S^- \quad \text{with} \quad S^\pm := \ker(\varepsilon \mp 1).$$

Consequently, the Dirac operator D decomposes as

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$$

with $D^\pm: \Gamma(X \setminus Z, S^\pm \otimes \mathbb{I}) \rightarrow \Gamma(X \setminus Z, S^\mp \otimes \mathbb{I})$ denoting the **chiral Dirac operators**.

The minimal and maximal extensions decompose analogously. Therefore, the Gelfand–Robbin quotient $\check{H}$ orthogonally decomposes as

$$\check{H} = \check{H}^+ \oplus \check{H}^- \quad \text{with} \quad \check{H}^\pm := \frac{\text{dom}(D_{\max}^\pm)}{\text{dom}(D_{\min}^\pm)};$$

moreover, $\check{H}^\pm \subseteq \check{H}$ are Lagrangian. A **chiral residue condition** is a closed subspace $R^\pm \subseteq \check{H}^\pm$. For every chiral residue condition $R^\pm \subseteq \check{H}^\pm$ there is a unique complementary chiral residue condition $R^\mp \subseteq \check{H}^\mp$ such that $R := R^+ \oplus R^- \subseteq \check{H}$ is a Lagrangian residue condition. As before, the chiral residue conditions $R^\pm \subseteq \check{H}^\pm$ precisely correspond to the closed extensions $D_{R^\pm}^\pm$ of $D_{\min}^\pm$.

The chirality operator also induces a parallel orthogonal decomposition $\check{S} = \check{S}^+ \oplus \check{S}^-$. With respect to it the branching locus operator A decomposes into $A^\pm := -JD_{\check{S}}^\pm: \Gamma(Z, \check{S}^\pm) \rightarrow \Gamma(Z, \check{S}^\pm)$; consequently,

$$\check{H}\Gamma(Z, \check{S}) = \check{H}\Gamma(Z, \check{S}^+) \oplus \check{H}\Gamma(Z, \check{S}^-).$$

The residue map res correspondingly decomposes into $\text{res}^\pm$, and these induce isomorphisms

$$\check{H}^\pm \cong \check{H}\Gamma(Z, \check{S}^\pm).$$

Of course, the conormal and adapted Sobolev spaces also decompose according to the chirality operator. In particular, for every chiral residue condition $R^\pm \subseteq \check{H}\Gamma(Z, \check{S}^\pm)$ and $k \in \mathbb{N}_0$ there are the following higher regularity versions of $D_{R^\pm}^\pm$:

$$D_{R^\pm}^{\pm, (k)} : H_a^{k+1}\Gamma(X \setminus Z, S^\pm \otimes \mathbb{I}; R^\pm) \rightarrow H_b^k\Gamma(X \setminus Z, S^\mp \otimes \mathbb{I}).$$

If the Lagrangian residue condition $R = R^+ \oplus R^-$ is elliptic, then both operators $D_{R^\pm}^{\pm, (k)}$ satisfy elliptic estimates and elliptic regularity; in particular, each is Fredholm.

A **local chiral residue condition** is a chiral residue condition of the form

$$R_{V^\pm}^\pm := \check{H}\Gamma(Z, V^\pm) \subseteq \check{H}\Gamma(Z, \check{S}^\pm)$$

for some subbundle $V^\pm \subseteq \check{S}^\pm$. For every subbundle $V^\pm \subseteq \check{S}^\pm$ there is a unique subbundle $V^\mp \subseteq \check{S}^\mp$ such that $V^+ \oplus V^- \subseteq \check{S} = \check{S}^+ \oplus \check{S}^-$ is Lagrangian. Of course, $V^\mp$ induces the complementary chiral residue condition: $R_V := R_{V^+}^+ \oplus R_{V^-}^-$ is Lagrangian. A moment's thought shows that R_V is elliptic if $V^\pm$ satisfies the symbolic criterion, namely for every $v \in TZ \setminus \{0\}$

$$V^\pm \oplus J_Y(v)V^\pm = \check{S}^\pm.$$

This holds in particular if $\text{rk}_{\mathbb{A}} \check{S}^\pm = 2$ and $V^\pm \subseteq \check{S}^\pm$ is an $\mathbb{A}$ -linear subbundle with $\text{rk}_{\mathbb{A}} V^\pm = 1$.

3 The universal Dirac operator and residue map

This section sets up the family version of the theory developed in [BW26] and surveyed in [Section 2](#). After describing **Dir**, the space of Dirac bundle structures on S , and **Ram**, the moduli space of ramified line bundles (with smooth branching locus) over X , the spaces $H_a^\infty\Gamma(X \setminus \text{Br}(\mathbb{I}), S \otimes \mathbb{I})$, $H_b^\infty\Gamma(X \setminus \text{Br}(\mathbb{I}), S \otimes \mathbb{I})$, and $\Gamma(\text{Br}(\mathbb{I}), \check{S})$ are assembled into tame Fréchet space bundles **E**, **F**, and **G** over $\text{Dir} \times \text{Ram}$. The Dirac operators and residue maps assemble into the universal Dirac operator $\mathbf{D} : \mathbf{E} \rightarrow \mathbf{F}$ and the universal residue map $\text{res} : \mathbf{E} \rightarrow \mathbf{G}$. These bundles are equipped with partial covariant derivatives, and the corresponding infinitesimal variation of $\mathbf{D}$ is determined. Within this framework it is possible to consider residue conditions in families. Finally, it is explained what additional structure arises from the chirality operator $\varepsilon := \gamma(\text{vol}_g)$ if X is oriented and $\dim X = 0 \pmod{4}$.

3.1 The space of Dirac bundles

For the purposes of this article, the following is the relevant parameter space and half of the base of the tame Fréchet space bundles **E**, **F**, and **G**.

Definition 3.1. Denote by $\mathbf{Met} \subseteq \Gamma(X, \text{Hom}(S^2TX, \mathbb{R}))$ the space of Riemannian metrics on X . The space of **Dirac bundle structures** on S is the tame Fréchet submanifold

$$\mathbf{Dir} := \left\{ (g; \gamma, \nabla) \in \mathbf{Met} \times \Gamma(X, \text{Hom}(TX, \mathfrak{o}(S))) \times \mathcal{A}(S) : \begin{array}{l} (\gamma, \nabla) \text{ is a Dirac bundle structure} \\ \text{with respect to } g \end{array} \right\}.$$

In the following a triple $(g; \gamma, \nabla) \in \mathbf{Dir}$ is often abbreviated to $\mathbf{p}$ (which, of course, stands for parameter). •

The projection map $p: \mathbf{Dir} \rightarrow \mathbf{Met}$ exhibits $\mathbf{Dir}$ as a tame smooth Fréchet fibre bundle over $\mathbf{Met}$. Bourguignon and Gauduchon [BG92] construct an Ehresmann connection on $\mathbf{Dir}$; see also [AWW16, §3.2; MN17, Definition 2.9; AD25, §2.4, §2.5]. Moreover, the vertical tangent bundle

$$T\mathbf{Dir}/\mathbf{Met} := \ker(Tp: T\mathbf{Dir} \rightarrow T\mathbf{Met})$$

further decomposes as follows.

Proposition 3.2. *For every $\mathbf{p} = (g; \gamma, \nabla) \in \mathbf{Dir}$ the tangent space $T_{\mathbf{p}}\mathbf{Dir}$ decomposes into the tame Fréchet subspaces*

$$T_{\mathbf{p}}\mathbf{Dir} = H_{\text{BG},\mathbf{p}} \oplus V_{\mathcal{G},\mathbf{p}} \oplus V_{\nabla,\text{Uh},\mathbf{p}}$$

defined as follows:

- (1) $H_{\text{BG},\mathbf{p}}$ is the horizontal distribution at $\mathbf{p} \in \mathbf{Dir}$ of the **Bourguignon–Gauduchon connection** [BG92]; that is: the unique Ehresmann connection on $\mathbf{Dir} \rightarrow \mathbf{Met}$ which horizontally lifts $\dot{g} \in T_g\mathbf{Met}$ to $(\dot{g}; \dot{\gamma}^{\text{BG}}, \dot{\nabla}^{\text{BG}}) \in T_{\mathbf{p}}\mathbf{Dir}$ defined by

$$\dot{\gamma}^{\text{BG}} := \frac{1}{2} \sum_{i=1}^d \dot{g}(-, e_i) \gamma(e_i) \quad \text{and} \quad \dot{\nabla}^{\text{BG}} := \frac{1}{8} \sum_{i,j=1}^d (\nabla_{e_i} \dot{g})(e_j, -) [\gamma(e_i), \gamma(e_j)].$$

Here and throughout $(e_1, \dots, e_d)$ denotes a local g -orthonormal frame.

- (2) $V_{\mathcal{G},\mathbf{p}}$ is generated by the action of the **gauge group** $\mathcal{G}(S) := \Gamma(X, \mathcal{O}(S))$ on $\mathbf{Dir}$; that is:

$$V_{\mathcal{G},\mathbf{p}} := \{(0; [\xi, \gamma], -\nabla \xi) : \xi \in \Gamma(X, \mathfrak{o}(S))\}.$$

- (3) $V_{\nabla,\text{Uh},\mathbf{p}}$ consists of infinitesimal variations of the spin connection in Uhlenbeck gauge; that is:

$$V_{\nabla,\text{Uh},\mathbf{p}} := \{(0; 0, \dot{\nabla}) : \dot{\nabla} \in \Omega^1(X, \mathfrak{o}_{\text{Cl}}(S)) \text{ satisfying } d_{\nabla}^* \dot{\nabla} = 0\}.$$

Here $\mathfrak{o}_{\text{Cl}}(S)$ denotes the bundle of skew-adjoint endomorphisms of S which commute with the Clifford multiplication γ .

In fact, the vertical tangent space $T_{\mathbf{p}}\mathbf{Dir}/\mathbf{Met}$ contains every infinitesimal variation of the spin connection; that is:

$$V_{\nabla,\mathbf{p}} := \{(0; 0, \dot{\nabla}) : \dot{\nabla} \in \Omega^1(X, \mathfrak{o}_{\text{Cl}}(S))\} \subseteq V_{\mathcal{G},\mathbf{p}} \oplus V_{\nabla,\text{Uh},\mathbf{p}}.$$

The above turns out to be useful in the following to organise various computations.

Proof of Proposition 3.2. To prove that H_{BG} is an Ehresmann connection on $\mathbf{Dir} \rightarrow \mathbf{Met}$, it suffices to verify the formulae

$$(3.3) \quad \dot{\gamma}^{\text{BG}}(v) \gamma(v) + \gamma(v) \dot{\gamma}^{\text{BG}}(v) = -\dot{g}(v, v) \quad \text{and}$$

$$(3.4) \quad [\dot{\nabla}_v^{\text{BG}}, \gamma(w)] = \dot{\gamma}^{\text{BG}}(\nabla_v w) + \gamma(\dot{\nabla}_v w) - [\nabla_v, \dot{\gamma}^{\text{BG}}(w)]$$

with $\dot{\nabla}$ denoting the infinitesimal deformation of the Levi-Civita connection corresponding to $\dot{g} \in T_g \mathbf{Met}$. The verification of (3.3) is straightforward from the formula in (1). By differentiating Koszul's formula,

$$2g(\dot{\nabla}_v w, u) = (\nabla_v \dot{g})(w, u) + (\nabla_w \dot{g})(u, v) - (\nabla_u \dot{g})(v, w).$$

Therefore,

$$\gamma(\dot{\nabla}_v w) = \frac{1}{2} \sum_{j=1}^d ((\nabla_v \dot{g})(w, e_j) + (\nabla_w \dot{g})(v, e_j) - (\nabla_{e_j} \dot{g})(v, w)) \gamma(e_j).$$

A direct computation using the identity

$$(3.5) \quad [[\gamma(u), \gamma(v)], \gamma(w)] = 4(\gamma(v)g(u, w) - \gamma(u)g(v, w))$$

yields

$$\begin{aligned} [\dot{\nabla}_v^{\text{BG}}, \gamma(w)] &= \frac{1}{2} \sum_{j=1}^d ((\nabla_w \dot{g})(e_j, v) - (\nabla_{e_j} \dot{g})(v, w)) \gamma(e_j), \\ \dot{\gamma}^{\text{BG}}(\nabla_v w) &= \frac{1}{2} \sum_{j=1}^d \dot{g}(\nabla_v w, e_j) \gamma(e_j), \quad \text{and} \\ [\nabla_v, \dot{\gamma}^{\text{BG}}(w)] &= \frac{1}{2} \sum_{j=1}^d ((\nabla_v \dot{g})(w, e_j) + \dot{g}(\nabla_v w, e_j)) \gamma(e_j). \end{aligned}$$

(3.4) follows directly from the preceding identities.

Evidently, $V_{\mathcal{G}, \mathbf{p}}$ and $V_{\nabla, \mathbf{p}}$ are contained in the vertical tangent bundle $T\mathbf{Dir}/\mathbf{Met}$. The Uhlenbeck gauge fixing condition in (3) ensures that $V_{\mathcal{G}, \mathbf{p}} \cap V_{\nabla, \text{Uh}, \mathbf{p}} = 0$. To prove that $V_{\mathcal{G}, \mathbf{p}} \oplus V_{\nabla, \text{Uh}, \mathbf{p}}$ is the entire vertical tangent space $T_p \mathbf{Dir}/\mathbf{Met}$, it suffices to show that every infinitesimal deformation $\dot{\gamma}$ of a Clifford multiplication γ is of the form $[\xi, \gamma]$ with $\xi \in \Gamma(X, \mathfrak{o}(S))$. This, however, is a consequence of the fact that Clifford algebras are separable finite-dimensional $\mathbb{R}$ -algebras and, therefore, their representations are infinitesimally rigid; see, e.g., [Hoc46, Theorem 5] or [Wei94, Lemma 9.1.9 and Theorem 9.2.11]. In fact, the arguments found loc. cit. reveal that $\dot{\gamma} = [\xi, \gamma]$ holds for

$$\xi := 2^{-d} \sum_{I \subseteq \{1, \dots, d\}} \xi_I \in \Gamma(X, \mathfrak{o}(S))$$

with

$$\xi_I := \sum_{\ell=1}^k (-1)^\ell \gamma(e_{i_1}) \cdots \gamma(e_{i_{\ell-1}}) \dot{\gamma}(e_{i_\ell}) \gamma(e_{i_\ell}) \gamma(e_{i_{\ell+1}}) \cdots \gamma(e_{i_k})$$

for $I = \{i_1, \dots, i_k\} \subseteq \{1, \dots, d\}$ with $i_1 < \dots < i_k$. Of course, this can also be verified by direct computation. $\blacksquare$

Remark 3.6. The $V_{\mathcal{G}, \mathbf{p}}$ and $V_{\nabla, \text{Uh}, \mathbf{p}}$ need not form subbundles because there can be $\mathbf{p} \in \mathbf{Dir}$ for which the map $\Gamma(X, \mathfrak{o}(S)) \rightarrow V_{\mathcal{G}, \mathbf{p}}$ has a non-trivial kernel. Of course, the $H_{\text{BG}, \mathbf{p}}$ and $V_{\nabla, \mathbf{p}}$ do form subbundles. $\clubsuit$

Remark 3.7. Here is how to understand the Lie algebra bundle $\mathfrak{o}_{Cl}(S)$. For every $x \in X$ and choice of orthogonal frame of $T_x X$, the fibre S_x is a representation of the Clifford algebra Cl_d . The representation S_x decomposes into irreducible representations, the **pinor representations** [Harg, Definition 11.10], which can be read off from the classification [Harg, Theorem 11.3]:

- (1) If $d = 0, 1, 2 \pmod{4}$, then there is a unique pinor representation $\mathbf{P}$ and, therefore, $S_x = \mathbf{P}^m$ for some $m \in \mathbb{N}_0$. The commuting algebra $\mathbb{K} := \text{End}_{Cl}(\mathbf{P})$ of $\mathbf{P}$ is isomorphic to $\mathbb{R}; \mathbb{C}; \mathbb{H}$ depending on whether $d = 0, 6; 1, 5; 2, 4 \pmod{8}$, respectively. Therefore,

$$\mathfrak{o}_{Cl}(S_x) \cong \begin{cases} \mathfrak{o}(m) & \text{if } d = 0, 6 \pmod{8} \\ \mathfrak{u}(m) & \text{if } d = 1, 5 \pmod{8} \\ \mathfrak{sp}(m) & \text{if } d = 2, 4 \pmod{8}. \end{cases}$$

- (2) If $d = 3 \pmod{4}$, then there are two pinor representations $\mathbf{P}^\pm$, distinguished by whether the volume element acts as ± 1 . Therefore, $S_x = (\mathbf{P}^+)^{m_+} \oplus (\mathbf{P}^-)^{m_-}$. The commuting algebra $\mathbb{K} := \text{End}_{Cl}(\mathbf{P}^\pm)$ of $\mathbf{P}^\pm$ is isomorphic to $\mathbb{H}, \mathbb{R}$, depending on whether $d = 3, 7 \pmod{8}$, respectively. Therefore,

$$\mathfrak{o}_{Cl}(S_x) \cong \begin{cases} \mathfrak{sp}(m_+) \oplus \mathfrak{sp}(m_-) & \text{if } d = 3 \pmod{8} \\ \mathfrak{o}(m_+) \oplus \mathfrak{o}(m_-) & \text{if } d = 7 \pmod{8}. \end{cases}$$

If X is spin and a spin structure $\mathfrak{s}$ on (X, g) is chosen, then the pinor representations give rise to pinor bundles $\mathbf{P}$ or $\mathbf{P}^+, \mathbf{P}^-$ over X and, consequently, $S = M \otimes_{\mathbb{K}} \mathbf{P}$ or $S = M^+ \otimes_{\mathbb{K}} \mathbf{P}^+ \oplus M^- \otimes_{\mathbb{K}} \mathbf{P}^-$ with M or M^+, M^- encoding the multiplicities. ♣

Remark 3.8. For every $(g, \gamma, \nabla) \in \text{Dir}$ the curvature $F_\nabla \in \Omega^2(X, \mathfrak{o}(S))$ decomposes into **Riemannian spin curvature** $\text{Riem}_g^S \in \Omega^2(X, \mathfrak{o}(S))$ and the **twisting curvature** $F_\nabla^{\text{tw}} \in \Omega^2(X, \mathfrak{o}_{Cl}(S))$ defined by

$$\begin{aligned} \text{Riem}_g^S(u, v) &:= \frac{1}{8} \sum_{k, \ell=1}^d \langle \text{Riem}_g(u, v) e_k, e_\ell \rangle [\gamma(e_k), \gamma(e_\ell)] \quad \text{and} \\ F_\nabla^{\text{tw}} &:= F_\nabla - \text{Riem}_g^S; \end{aligned}$$

see [BGV92, Proposition 3.43]. This decomposition is useful, because the contribution of Riem_g^S to various formulae often simplifies quite notably. ♣

3.2 The space of ramified Euclidean line bundles

Here is the other half of the base of the tame Fréchet space bundles $\mathbf{E}, \mathbf{F}$, and $\mathbf{G}$.

Definition 3.9 (cf. [DW24, §3.4.3]). The **moduli space of ramified Euclidean line bundles** over X is constructed as follows:

- (1) Denote by **Sub** the set of closed codimension-two submanifolds $Z \subseteq X$. Equip **Sub** with the following tame Fréchet manifold structure described in [Ham82, Part I Example 4.1.7 and Example 4.4.7, and Part II Corollary 2.3.7].

Given $Z \in \mathbf{Sub}$ and a tubular neighbourhood $J: NZ \supseteq U \rightarrow X$ of Z , define the map $\phi_J^{-1}: \Gamma(Z, NZ) \supseteq \Gamma(Z, U) \rightarrow \mathbf{Sub}$ by

$$\phi_J^{-1}(v) := \text{im}(J \circ v).$$

Denote by ϕ_J the inverse of $\phi_J^{-1}: \Gamma(Z, U) \rightarrow \text{im } \phi_J^{-1}$. Equip $\Gamma(Z, NZ)$ with the usual structure of a tame Fréchet space; see [Ham82, Corollary II.1.3.9]. The charts constructed in [Ham82, Example I.4.1.7] are of the form ϕ_J .

- (2) Denote by **Ram** the set of isometry classes of ramified Euclidean line bundles $(Z, \mathbf{I})$ over X with smooth Z . Consider the map

$$\text{Br}: \mathbf{Ram} \rightarrow \mathbf{Sub}$$

assigning to $(Z, [\mathbf{I}])$ the branching locus Z .

- (3) For every open subset $U \subseteq X$ denote by $\mathbf{Sub}_U \subseteq \mathbf{Sub}$ the open subset of those $Z \in \mathbf{Sub}$ for which:
- (a) $Z \subseteq U$, and
 - (b) the restriction $H^1(X \setminus Z, \{\pm 1\}) \rightarrow H^1(X \setminus U, \{\pm 1\})$ is an isomorphism.

The subset $\mathbf{Sub}_U$ is open and the monodromy representations assemble into an injection

$$\tau_U: \text{Br}^{-1}(\mathbf{Sub}_U) \rightarrow \mathbf{Sub}_U \times H^1(X \setminus U, \{\pm 1\}).$$

Equip **Ram** with the coarsest topology with respect to which the maps τ_U are continuous.

- (4) With respect to the above topology $\text{Br}: \mathbf{Ram} \rightarrow \mathbf{Sub}$ is a locally finite covering map; cf. [tDieo8, proof of Theorem 3.3.2]. In particular, **Ram** inherits the structure of a tame Fréchet manifold from **Sub**. Moreover, for every $(Z, [\mathbf{I}]) \in \mathbf{Ram}$,

$$T_{(Z, [\mathbf{I}])} \mathbf{Ram} = T_Z \mathbf{Sub} = \Gamma(Z, NZ).$$

In the following $(Z, [\mathbf{I}])$ is frequently abbreviated to $[\mathbf{I}]$. This is harmless, because Z can be recovered as $\text{Br}(\mathbf{I})$. •

Remark 3.10. Here are some further comments regarding (3). Euclidean line bundles $\mathbf{I}$ over $X \setminus Z$ are classified by their Stiefel–Whitney class $w_1(\mathbf{I}) \in \text{Hom}(\pi_1(X \setminus Z), \{\pm 1\}) = H^1(X \setminus Z, \{\pm 1\})$. The latter cohomology group fits into the exact sequence

$$H^1(X; \{\pm 1\}) \xrightarrow{i^*} H^1(X \setminus Z; \{\pm 1\}) \xrightarrow{\delta} H^2(X, X \setminus Z; \{\pm 1\}) \xrightarrow{j^*} H^2(X, \{\pm 1\}).$$

Excision and the Thom isomorphism identify $H^2(X, X \setminus Z; \{\pm 1\}) \cong H^0(Z, \{\pm 1\})$. The monodromy condition means that $\delta(w_1(\mathbf{I})) = -1 \in H^0(Z, \{\pm 1\})$. In particular, its image under the final map vanishes. But this is exactly the Poincaré dual of $[Z] \in H_{d-2}(X, \{\pm 1\})$. Therefore, the latter must vanish for Z to arise as the branching locus; moreover, if it vanishes, then the space of isometry classes of ramified Euclidean line bundles with branching locus Z is an $H^1(X, \{\pm 1\})$ -torsor. ♣

Proposition 3.11 (cf. [DW24, Proposition 3.60]). *Ram carries a **twisted universal ramified Euclidean line bundle** in the following sense:*

- (1) Consider the **universal non-branching locus**

$$\mathring{X} := \{([I], x) \in \mathbf{Ram} \times X : x \notin \mathrm{Br}(I)\} \subseteq \mathbf{Ram} \times X.$$

The projection map $p: \mathring{X} \rightarrow \mathbf{Ram}$ is a fibre bundle.

- (2) Denote by $\mathfrak{U}$ the set of those open subsets $\mathcal{U} \subseteq \mathbf{Ram}$ for which $p: \mathring{X}|_{\mathcal{U}} \rightarrow \mathcal{U}$ is trivial. For every $\mathcal{U} \in \mathfrak{U}$ there is a Euclidean line bundle $\mathfrak{L}_{\mathcal{U}}$ over $\mathring{X}|_{\mathcal{U}}$ and for every $[I] \in \mathcal{U}$ an isomorphism

$$\mathfrak{L}_{\mathcal{U}}|_{[I] \times (X \setminus \mathrm{Br}(I))} \cong \mathbb{I}$$

unique up to $\mathrm{Aut}(\mathbb{I}) = \{\pm 1\}$.

- (3) There is a Čech 2-cocycle $\lambda \in \check{C}^2(\mathfrak{U}, \{\pm 1\})$ together with, for every $\mathcal{U}, \mathcal{V} \in \mathfrak{U}$, an isomorphism $\phi_{\mathcal{V}}^{\mathcal{U}}: \mathfrak{L}_{\mathcal{U}}|_{\mathring{X}|_{\mathcal{U} \cap \mathcal{V}}} \cong \mathfrak{L}_{\mathcal{V}}|_{\mathring{X}|_{\mathcal{U} \cap \mathcal{V}}}$ such that for every $\mathcal{U}, \mathcal{V}, \mathcal{W} \in \mathfrak{U}$

$$p^* \lambda_{\mathcal{U}\mathcal{V}\mathcal{W}} := \phi_{\mathcal{U}}^{\mathcal{W}} \phi_{\mathcal{V}}^{\mathcal{U}} \phi_{\mathcal{W}}^{\mathcal{U}} \in \check{C}^0(\mathring{X}|_{\mathcal{U} \cap \mathcal{V} \cap \mathcal{W}}, \{\pm 1\}).$$

Here $\{\pm 1\}$ is the constant sheaf associated with the abelian group $\{\pm 1\}$.

Proof. The proof is identical to that of [DW24, Proposition 3.60] and, therefore, omitted. $\blacksquare$

The Čech cohomology class $[\lambda] \in \check{H}^2(\mathbf{Ram}, \{\pm 1\})$ is the obstruction to gluing $\{\mathfrak{L}_{\mathcal{U}} : \mathcal{U} \in \mathfrak{U}\}$ to an untwisted universal ramified Euclidean line bundle $\mathfrak{L}$ over $\mathring{X}$. Unfortunately, it does not always vanish. A pragmatic solution to this problem is to instead work with the Euclidean line bundle

$$\mathfrak{L} := \bigsqcup_{\mathcal{U} \in \mathfrak{U}} \mathfrak{L}_{\mathcal{U}} \quad \text{and} \quad \mathbf{Ram}^0 := \bigsqcup_{\mathcal{U} \in \mathfrak{U}} \mathcal{U}.$$

The significance of $\mathfrak{L}$ is that for every $(\mathcal{U}; [I]) \in \mathbf{Ram}^0$ it selects a choice of representative of $[I]$. Henceforth, this choice shall be denoted by I , abusing but simplifying notation. Moreover, in the upcoming discussion, $(\mathcal{U}; [I])$ is abbreviated to $[I]$ and $\mathbf{Ram}^0$ to $\mathbf{Ram}$. This abuse of notation is ultimately inconsequential, because ultimately $\mathbb{Z}/2\mathbb{Z}$ harmonic spinors are considered up to the action of $\mathbb{R}^\times$ and $\{\pm 1\} \leq \mathbb{R}^\times$: forming the quotient by $\{\pm 1\}$ removes the ambiguity.

Remark 3.12. The origin of the above issue is the stabiliser $\mathrm{Aut}(I) = \{\pm 1\}$ and, in principle, one ought to work with orbifolds to deal with it. The Čech 2-cocycle λ produces a proper étale tame Fréchet Lie groupoid which presents a tame Fréchet orbifold $\mathbf{Ram}$: the **fine moduli space of ramified Euclidean line bundles**; cf. [Moe02, §3]. The corresponding orbifold counterpart $\mathring{X}$ of X does admit a **universal ramified Euclidean line bundle** $\mathfrak{L}$; cf. [Moe02, §5]. $\clubsuit$

3.3 The extended gauge group and the Kosmann lift

The following symmetries of $\mathbf{Dir} \times \mathbf{Ram}$ are crucial for the upcoming discussion.

Definition 3.13. The **extended gauge group** of S is the tame Fréchet Lie group $\widehat{\mathcal{G}}(S)$ constructed as follows:

- (1) An **extended isometry** of S is a diffeomorphism $u \in \text{Diff}(S)$ for which there exists a diffeomorphism $\check{u} \in \text{Diff}(X)$ such that u defines an isometry $S \cong \check{u}^*S$. These form a subgroup $\mathcal{G}(S) \leq \text{Diff}(S)$.
- (2) A moment's thought shows that extended isometries lift to $O(r)$ -equivariant diffeomorphisms of $\text{Fr}(S)$, the orthogonal frame bundle of S ; in fact, $\widehat{\mathcal{G}}(S) = \text{Diff}(\text{Fr}(S))^{O(r)}$, abusing notation. By [Molio, Proposition 2.6], $\widehat{\mathcal{G}}(S) \leq \text{Diff}(\text{Fr}(S))$ is a tame Fréchet Lie subgroup.
- (3) From the above perspective, $\widehat{\mathcal{G}}(S)$ acts smoothly and tamely on $S = \text{Fr}(S) \times_{O(r)} \mathbb{R}^r$ and $X = \text{Fr}(S) \times_{O(r)} \{*\}$ from the left by

$$u \cdot [p, \phi] := [u(p), \phi] \quad \text{and} \quad u \cdot [p, *] := [u(p), *],$$

respectively. As a consequence, it acts smoothly and tamely on $\Gamma(X, S)$, $\mathbf{Dir}$, and $\mathbf{Ram}$ from the right by pullback.¹ •

The subset $\text{Diff}_S(X) \subseteq \text{Diff}(X)$ of those diffeomorphisms that lift to $\widehat{\mathcal{G}}(S)$ forms an open subgroup. In fact, $\widehat{\mathcal{G}}(S)$ fits into the exact sequence of tame Fréchet Lie groups

$$\mathcal{G}(S) \hookrightarrow \widehat{\mathcal{G}}(S) \rightarrow \text{Diff}_S(X).$$

This induces a corresponding exact sequence of tame Fréchet Lie algebras

$$\Gamma(X, \mathfrak{o}(S)) \hookrightarrow \text{Lie}(\widehat{\mathcal{G}}(S)) := T_1\widehat{\mathcal{G}}(S) \rightarrow \text{Vect}(X).$$

As a sequence of tame Fréchet *spaces*, this splits, but not canonically so. For example, for every orthogonal connection ∇ on S the **horizontal lift**

$$\text{lift}_\nabla : \text{Vect}(X) \rightarrow \text{Lie}(\widehat{\mathcal{G}}(S)) \subseteq \text{Vect}(S)$$

defines a right splitting. The right Lie algebra action of $\text{lift}_\nabla(v)$ on $\mathbf{p} = (g; \gamma, \nabla) \in \mathbf{Dir}$ yields $(\mathcal{L}_v g; \dot{\gamma}, \dot{\nabla}) \in T_{\mathbf{p}}\mathbf{Dir}$ with $\dot{\gamma}(w) := \gamma(\nabla_w v)$. For the purposes of this article it is convenient to use the following lift that fits better with the Bourguignon–Gauduchon connection defined in Proposition 3.2 (1).

Definition 3.14 (cf. [Kos72, Definition III.2.1; BG92, Proposition 17]). For every $\mathbf{p} = (g; \gamma, \nabla) \in \mathbf{Dir}$ the **Kosmann lift** is the tame linear map $\text{lift}_{\mathbf{p}}^K : \text{Vect}(X) \rightarrow \text{Lie}(\widehat{\mathcal{G}}(S))$ defined by

$$\text{lift}_{\mathbf{p}}^K(v) := \text{lift}_\nabla(v) + \kappa_v \quad \text{with}$$

$$\kappa_v := -\frac{1}{8} \sum_{i,j=1}^d g(\nabla_{e_i} v, e_j) [\gamma(e_i), \gamma(e_j)] \in \Gamma(X, \mathfrak{o}(S)).$$

¹ The reader should be warned that $\widehat{\mathcal{G}}(S)$ does not lift to an action on $\mathbf{Ram}^0$; however, the action lifts locally. This is sufficient for Proposition 3.22.

Proposition 3.15 (cf. [Kos72, §V.1.2]). *Let $\mathbf{p} \in \mathbf{Dir}$ and $v \in \mathbf{Vect}(X)$.*

- (1) *The right Lie algebra action of $\text{lift}_{\mathbf{p}}^K(v) \in \text{Lie}(\widehat{\mathcal{G}}(S))$ on $\phi \in \Gamma(X, S)$ is given by the **Kosmann Lie derivative**; that is:*

$$\phi \cdot \text{lift}_{\mathbf{p}}^K(v) = \mathcal{L}_v^K \phi \quad \text{with} \quad \mathcal{L}_v^K := \nabla_v + \kappa_v.$$

- (2) *The right Lie algebra action of $\text{lift}_{\mathbf{p}}^K(v) \in \text{Lie}(\widehat{\mathcal{G}}(S))$ on $\mathbf{p} \in \mathbf{Dir}$ is related to the Bourguignon–Gauduchon lift $(\dot{g}; \dot{\gamma}^{\text{BG}}, \dot{\nabla}^{\text{BG}}) \in T_{\mathbf{p}}\mathbf{Dir}$ of $\dot{g} := \mathcal{L}_v g \in T_g\mathbf{Met}$, defined in [Proposition 3.2 \(1\)](#), by*

$$(3.16) \quad \mathbf{p} \cdot \text{lift}_{\mathbf{p}}^K(v) = (\dot{g}; \dot{\gamma}^{\text{BG}}, \dot{\nabla}^{\text{BG}} + F_{\nabla}^{\text{tw}}(v, -)).$$

Remark 3.17. The following should help to make the Kosmann lift more palatable:

- (1) The requirement that the right Lie algebra action of the lift $\tilde{v} \in \text{Lie}(\widehat{\mathcal{G}}(S))$ of $v \in \mathbf{Vect}(X)$ reproduces $\dot{\gamma}^{\text{BG}}$ determines $\tilde{v}$ up to the addition of an arbitrary $\xi_v \in \Gamma(X, \mathfrak{o}_{\text{Cl}}(S))$.
- (2) The requirement (3.16) determines $\text{lift}_{\mathbf{p}}^K(v)$ uniquely up to the addition of an arbitrary $\xi_v \in \ker(\nabla : \Gamma(X, \mathfrak{o}_{\text{Cl}}(S)) \rightarrow \Omega^1(X, \mathfrak{o}_{\text{Cl}}(S)))$. The latter vanishes for typical $\mathbf{p} \in \mathbf{Dir}$. Therefore, (3.16) and a suitable continuity condition determine the Kosmann lift uniquely.
- (3) For generic $\mathbf{p} \in \mathbf{Dir}$ and $v \in \mathbf{Vect}(X)$, $F_{\nabla}^{\text{tw}}(v, -) \notin \text{im}(\nabla : \Gamma(X, \mathfrak{o}_{\text{Cl}}(S)) \rightarrow \Omega^1(X, \mathfrak{o}_{\text{Cl}}(S)))$. As a consequence, this term cannot be eliminated from (3.16).
- (4) The reader should be warned that, despite its name, the Kosmann Lie derivative does not define a Lie algebra homomorphism $\mathbf{Vect}(X) \rightarrow \text{Lie}(\widehat{\mathcal{G}}(S))$; see [BG92, Proposition 18]. It does not even agree with the Lie derivative on $\Omega^\bullet(X)$ for the Dirac bundle that gives rise to the Hodge–de Rham operator $d + d^*$. ♣

Proof of Proposition 3.15. (1) is obvious. From this (2) is derived by the following direct computation. Set

$$(\dot{g}; \dot{\gamma}, \dot{\nabla}) := \mathbf{p} \cdot \text{lift}_{\mathbf{p}}^K(v).$$

Evidently, $\dot{g} = \mathcal{L}_v g$. Since

$$(3.18) \quad (\mathcal{L}_v g)(w, z) = g(\nabla_w v, z) + g(\nabla_z v, w),$$

the formulae in [Proposition 3.2 \(1\)](#) concretise to

$$\begin{aligned} \dot{\gamma}^{\text{BG}}(w) &= \frac{1}{2} \sum_{i=1}^d (g(\nabla_w v, e_i) + g(\nabla_{e_i} v, w)) \gamma(e_i) \quad \text{and} \\ \dot{\nabla}_w^{\text{BG}} &= \frac{1}{8} \sum_{i,j=1}^d (g(\nabla_{e_i}^2 v, w) + g(\nabla_{e_i, w}^2 v, e_j)) [\gamma(e_i), \gamma(e_j)]. \end{aligned}$$

Therefore, using (3.5),

$$\begin{aligned}
\dot{\gamma}(w) &= [\mathcal{L}_v^K, \gamma(w)] - \gamma([v, w]) = [\kappa_v, \gamma(w)] + \gamma(\nabla_w v) \\
&= -\frac{1}{8} \sum_{i,j=1}^d g(\nabla_{e_i} v, e_j) [[\gamma(e_i), \gamma(e_j)], \gamma(w)] + \gamma(\nabla_w v) \\
&= \frac{1}{2} \sum_{i=1}^d (g(\nabla_w v, e_i) + g(\nabla_{e_i} v, w)) \gamma(e_i) = \dot{\gamma}^{\text{BG}}(w).
\end{aligned}$$

Moreover,

$$\begin{aligned}
\dot{\nabla}_w &= [\mathcal{L}_v^K, \nabla_w] - \nabla_{[v, w]} = [\nabla_v, \nabla_w] - \nabla_{[v, w]} - [\nabla_w, \kappa_v] \\
&= \text{Riem}_g^S(v, w) + F_{\nabla}^{\text{tw}}(v, w) + \frac{1}{8} \sum_{i,j=1}^d g(\nabla_{w, e_i}^2 v, e_j) [\gamma(e_i), \gamma(e_j)] \\
&= \dot{\nabla}_w^{\text{BG}} + F_{\nabla}^{\text{tw}}(v, w) + R
\end{aligned}$$

with R denoting the remainder

$$R := \frac{1}{8} \sum_{i,j=1}^d (g(\text{Riem}_g(v, w) e_i, e_j) + g(\nabla_{w, e_i}^2 v, e_j) - g(\nabla_{e_i, e_j}^2 v, w) - g(\nabla_{e_i, w}^2 v, e_j)) [\gamma(e_i), \gamma(e_j)].$$

As a consequence of the algebraic Bianchi identity,

$$\begin{aligned}
R &= \frac{1}{16} \sum_{i,j=1}^d (g(\text{Riem}_g(v, w) e_i, e_j) + 2g(\text{Riem}_g(w, e_i) v, e_j)) [\gamma(e_i), \gamma(e_j)] \\
&= \frac{1}{16} \sum_{i,j=1}^d (g(\text{Riem}_g(v, w) e_i, e_j) + g(\text{Riem}_g(v, e_j) w, e_i) + g(\text{Riem}_g(v, e_i) e_j, w)) [\gamma(e_i), \gamma(e_j)] \\
&= 0.
\end{aligned}$$

■

3.4 The universal bundles of (admissible) singular spinors and residues

The following discussion constructs $\mathbf{E}$, $\mathbf{F}$, and $\mathbf{G}$ as well as $\mathbf{D}$ and $\mathbf{res}$ using the method introduced by Donaldson [Don21a, §4].

Definition 3.19. Here is the first stage of the construction:

- (1) The universal bundle of admissible singular spinors is the set

$$\mathbf{E} := \bigsqcup_{(\mathbf{p}, [\mathbf{I}]) \in \mathbf{Dir} \times \mathbf{Ram}} H_a^\infty \Gamma(X \setminus \text{Br}(\mathbf{I}), S \otimes \mathbf{I}; \mathbf{p})$$

together with the projection map $\mathbf{E} \rightarrow \mathbf{Dir} \times \mathbf{Ram}$ and the tame Fréchet space structures on the fibres, defined in Section 2.3 and [BW26, Definition 4.2 (b)]. Here and throughout the remainder of this article, additional decorations with $\mathbf{p}$ and $\mathbf{I}$ are used to indicate the dependence on these parameters.

- (2) The **universal bundle of singular spinors** is the set

$$\mathbf{F} := \coprod_{(\mathbf{p}, [\mathbf{I}]) \in \mathbf{Dir} \times \mathbf{Ram}} H_b^\infty \Gamma(X \setminus \text{Br}(\mathbf{I}), S \otimes \mathbf{I}; \mathbf{p})$$

together with the projection map $\mathbf{F} \rightarrow \mathbf{Dir} \times \mathbf{Ram}$ and the tame Fréchet space structures on the fibres, defined in [Section 2.3](#) and [\[BW26, Definition 4.2 \(a\)\]](#).

- (3) The **universal bundle of residues** is the set

$$\mathbf{G} := \coprod_{(\mathbf{p}, [\mathbf{I}]) \in \mathbf{Dir} \times \mathbf{Ram}} \Gamma(\text{Br}(\mathbf{I}), \check{S}_{\mathbf{p}})$$

together with the projection map $\mathbf{G} \rightarrow \mathbf{Dir} \times \mathbf{Ram}$ and the usual tame Fréchet space structures on the fibres.

- (4) The **universal Dirac operator** $\mathbf{D}: \mathbf{E} \rightarrow \mathbf{F}$ is defined by

$$\mathbf{D}(\mathbf{p}, [\mathbf{I}]; \phi) := (\mathbf{p}, [\mathbf{I}]; D_{\mathbf{p}}^{\mathbf{I}} \phi).$$

- (5) The **universal residue map** $\mathbf{res}: \mathbf{E} \rightarrow \mathbf{G}$ is defined by

$$\mathbf{res}(\mathbf{p}, [\mathbf{I}]; \phi) := (\mathbf{p}, [\mathbf{I}]; \text{res } \phi).$$

It remains to endow $\mathbf{E}$, $\mathbf{F}$, and $\mathbf{G}$ with the structure of tame Fréchet space bundles such that $\mathbf{D}$ and $\mathbf{res}$ become maps of tame Fréchet space bundles. •

$$\begin{array}{ccc} \mathbf{E} & \xrightarrow{\mathbf{D}} & \mathbf{F} \\ \text{res} \downarrow & & \\ \mathbf{G} & & \end{array}$$

Figure 1: The universal Dirac operator and residue map.

The above task is not entirely trivial. Even if $(Z_0, [\mathbf{I}_0]) \in \mathbf{Ram}$ is held fixed and $\mathbf{p}_0, \mathbf{p} \in \mathbf{Dir}$ are close, $H_a^\infty(X \setminus Z_0, S \otimes \mathbf{I}_0; \mathbf{p}_0)$ and $H_a^\infty(X \setminus Z_0, S \otimes \mathbf{I}_0; \mathbf{p})$ need not agree, even as subspaces of $\Gamma(X \setminus Z_0, S \otimes \mathbf{I}_0)$, unless a much stronger relation between $\mathbf{p}_0$ and $\mathbf{p}$ is assumed.

Definition 3.20. A set of **normal data** for a codimension two submanifold $Z_0 \subseteq X$ consists of:

- (N1) a rank 2 subbundle $N \subseteq TX|_{Z_0}$ complementary to TZ_0 ,
- (N2) a Euclidean metric h on N , and
- (N3) an $I \in \Gamma(Z_0, \text{Hom}(\Lambda^2 N, \text{End}(S|_{Z_0})))$ such that $I(\text{or}) \in \text{End}(S_x)$ is an orthogonal complex structure for every $x \in Z_0$ and $\text{or} \in \Lambda^2 N_x$ with $|\text{or}| = 1$.

Every $\mathbf{p} = (g; \gamma, \nabla) \in \mathbf{Dir}$ defines normal data for a codimension two submanifold $Z \subseteq X$ by declaring

$$N := TZ^\perp, \quad h := g|_N, \quad \text{and} \quad I := \gamma|_{\Lambda^2 N}.$$

The **slice** associated with a ramified Euclidean line bundle $(Z_0, [I_0]) \in \mathbf{Ram}$ and normal data (N, h, I) for Z_0 is the tame Fréchet submanifold

$$\mathbf{Sl} := \{(\mathbf{p}, [I_0]) \in \mathbf{Dir} \times \mathbf{Ram} : \mathbf{p} \text{ induces the normal data } (N, h, I) \text{ for } Z_0\} \subseteq \mathbf{Dir} \times \mathbf{Ram}.$$

These slices foliate $\mathbf{Dir} \times \mathbf{Ram}$ in the sense that every $(\mathbf{p}, [I]) \in \mathbf{Dir} \times \mathbf{Ram}$ is contained in some slice. •

Proposition 3.21. *Let $\mathbf{Sl} \subseteq \mathbf{Dir} \times \mathbf{Ram}$ be a slice and $(\mathbf{p}_0, [I_0]) \in \mathbf{Sl}$.*

(1) *For every $(\mathbf{p}, [I]) \in \mathbf{Sl}$ the identity maps induce tame isomorphisms*

$$\begin{aligned} H_a^\infty \Gamma(X \setminus \text{Br}(I_0), S \otimes I_0; \mathbf{p}_0) &\cong H_a^\infty \Gamma(X \setminus \text{Br}(I), S \otimes I; \mathbf{p}) = \mathbf{E}_{\mathbf{p}, I}, \\ H_b^\infty \Gamma(X \setminus \text{Br}(I_0), S \otimes I_0; \mathbf{p}_0) &\cong H_b^\infty \Gamma(X \setminus \text{Br}(I), S \otimes I; \mathbf{p}) = \mathbf{F}_{\mathbf{p}, I}, \quad \text{and} \\ \Gamma(\text{Br}(I_0), \check{S}_{\mathbf{p}_0}) &\cong \Gamma(\text{Br}(I), \check{S}_{\mathbf{p}}) = \mathbf{G}_{\mathbf{p}, I}. \end{aligned}$$

(2) *The maps*

$$\begin{aligned} \mathbf{D}|_{\mathbf{Sl}} : \mathbf{Sl} \times H_a^\infty \Gamma(X \setminus \text{Br}(I_0), S \otimes I_0; \mathbf{p}_0) &\rightarrow H_b^\infty \Gamma(X \setminus \text{Br}(I_0), S \otimes I_0; \mathbf{p}_0) \quad \text{and} \\ \text{res}|_{\mathbf{Sl}} : \mathbf{Sl} \times H_a^\infty \Gamma(X \setminus \text{Br}(I_0), S \otimes I_0; \mathbf{p}_0) &\rightarrow \Gamma(\text{Br}(I_0), \check{S}_{\mathbf{p}_0}) \end{aligned}$$

induced by $\mathbf{D}$ and res are tame smooth.

Proof. By [BW26, Proposition 3.14], $D_0 := D_{\mathbf{p}_0}^{I_0}$ and $D_{\mathbf{p}}^{I_0}$ differ from (cut-offs of) the corresponding model operators $\check{D}_{\mathbf{p}_0}^{I_0}$ and $\check{D}_{\mathbf{p}}^{I_0}$ [BW26, Definition 3.12] by operators in $\text{DiffOp}_b^1(S \otimes I_0)$. Direct inspection of the model operators and a glance at [BW26, Proof of Lemma 4.29] show that

$$D_{\mathbf{p}}^{I_0} = D_{\mathbf{p}_0}^{I_0} + B_{\mathbf{p}, \mathbf{p}_0}$$

with $B_{\mathbf{p}, \mathbf{p}_0} \in \text{DiffOp}_b^1(S \otimes I_0)$. This immediately implies (1). Inspection of $B_{\mathbf{p}, \mathbf{p}_0}$ establishes (2) as in [Ham82, Part II Theorem 3.3.3]. ■

This provides the desired tame Fréchet bundle structures for the restrictions $\mathbf{E}|_{\mathbf{Sl}}$, $\mathbf{F}|_{\mathbf{Sl}}$, and $\mathbf{G}|_{\mathbf{Sl}}$ to slices: in fact, it canonically trivialises them. These tame Fréchet bundle structures can be extended with the help of the following method due to Donaldson [Don21a].

Proposition 3.22 (cf. [Don21a, Proposition 4.2]). *Let $\mathbf{Sl}$ be a slice. For every $(\mathbf{p}_0, [I_0]) \in \mathbf{Sl}$ there is a **local uniformiser**; that is: an open neighbourhood $\mathcal{U} \subseteq \mathbf{Dir} \times \mathbf{Ram}$ of $(\mathbf{p}_0, [I_0])$ and a tame smooth map*

$$u : \mathcal{U} \rightarrow \widehat{\mathcal{G}}(S)$$

such that for every $(\mathbf{p}, [I]) \in \mathcal{U}$

$$u(\mathbf{p}, [I])^*(\mathbf{p}, [I]) \in \mathbf{Sl}.$$

Moreover, given a compact subset $K \subseteq X \setminus \text{Br}(I_0)$, the local uniformiser can be chosen such that $u(\mathbf{p}, [I])|_K = 1_S$ for every $(\mathbf{p}, [I]) \in \mathcal{U}$.

Proof. [Don21a, Proof of Proposition 4.2] explains the construction of a tame smooth map $\check{u}: \mathcal{U} \rightarrow \text{Diff}_0(X)$ which admits a lift $u: \mathcal{U} \rightarrow \widehat{\mathcal{G}}(S)$ such that, for every $(\mathbf{p}, [I]) \in \mathcal{U}$, $u(\mathbf{p}, [I])|_K = 1_S$ and $u(\mathbf{p}, [I])^* \mathbf{p}$ defines the same normal data as $\mathbf{p}_0$ —except possibly for the almost complex structure I , because in [Don21a] there is no analogue of (N3). After possibly shrinking $\mathcal{U}$, this defect can be repaired by observing the following:

- (1) For every I as in (N3) that is sufficiently close to I_0 , there is a $u_0 \in \mathcal{G}(S|_{Z_0})$ close to $1_{S|_{Z_0}}$ with $u_0^* I = I_0$;
- (2) Since u_0 is close to $1_{S|_{Z_0}}$, it can be extended to $\bar{u}_0 \in \mathcal{G}(S)$.

Evidently, this repair can be carried out in a way that produces the desired tame smooth map $u: \mathcal{U} \rightarrow \widehat{\mathcal{G}}(S)$. ■

The following two observations are verified by direct inspection.

Proposition 3.23. *Let $(\mathbf{p}, [I]) \in \text{Dir} \times \text{Ram}$. Every extended gauge transformation $u \in \widehat{\mathcal{G}}(S)$ induces tame isomorphisms*

$$\begin{aligned} u: H_a^\infty \Gamma(X \setminus \text{Br}(I), S \otimes I; \mathbf{p}) &\rightarrow H_a^\infty (X \setminus \text{Br}(\check{u}^* I), S \otimes \check{u}^* I; u^* \mathbf{p}), \\ u: H_b^\infty \Gamma(X \setminus \text{Br}(I), S \otimes I; \mathbf{p}) &\rightarrow H_b^\infty (X \setminus \text{Br}(\check{u}^* I), S \otimes \check{u}^* I; u^* \mathbf{p}), \quad \text{and} \\ u: \Gamma(\text{Br}(I), \check{S}_{\mathbf{p}}) &\rightarrow \Gamma(\text{Br}(\check{u}^* I), \check{S}_{u^* \mathbf{p}}). \end{aligned}$$

In particular, these lift the action of $\widehat{\mathcal{G}}(S)$ on the right of $\text{Dir} \times \text{Ram}$ to $\mathbf{E}, \mathbf{F}$, and $\mathbf{G}$. ■

Proposition 3.24. *Let Sl be a slice and $(\mathbf{p}_0, [I_0]) \in \text{Sl}$.*

- (1) *The subgroup*

$$\text{Stab}_{\widehat{\mathcal{G}}(S)}(\text{Sl}) := \{u \in \widehat{\mathcal{G}}(S) : u^* \text{Sl} = \text{Sl}\} \subset \widehat{\mathcal{G}}(S)$$

is a tame Fréchet Lie subgroup.

- (2) *The right actions of $\text{Stab}_{\widehat{\mathcal{G}}(S)}(\text{Sl})$ on $H_a^\infty(X \setminus Z_0, S \otimes I_0; \mathbf{p}_0)$, $H_b^\infty(X \setminus Z_0, S \otimes I_0; \mathbf{p}_0)$, and $\Gamma(Z_0, \check{S}_{\mathbf{p}_0})$ are tame smooth.* ■

Definition 3.25. The following completes the construction begun in Definition 3.19. Endow $\mathbf{E}, \mathbf{F}$, and $\mathbf{G}$ with the unique tame smooth Fréchet bundle structure in which for every $(\mathbf{p}_0, [I_0]) \in \text{Dir} \times \text{Ram}$ and for every local uniformiser $u: \mathcal{U} \rightarrow \widehat{\mathcal{G}}(S)$ as in Proposition 3.22 the bijections $\sigma: \mathbf{E}|_{\mathcal{U}} \rightarrow \mathcal{U} \times H_a^\infty(X \setminus \text{Br}(I_0), S \otimes I_0; \mathbf{p}_0)$, $\tau: \mathbf{F}|_{\mathcal{U}} \rightarrow \mathcal{U} \times H_b^\infty(X \setminus \text{Br}(I_0), S \otimes I_0; \mathbf{p}_0)$, and $v: \mathbf{G}|_{\mathcal{U}} \rightarrow \mathcal{U} \times \Gamma(\text{Br}(I_0), \check{S}_{\mathbf{p}_0})$ defined by

$$\begin{aligned} \sigma(\mathbf{p}, [I]; \phi) &:= (\mathbf{p}, [I]; u(\mathbf{p}, [I])\phi), \quad \tau(\mathbf{p}, [I]; \phi) := (\mathbf{p}, [I]; u(\mathbf{p}, [I])\phi), \quad \text{and} \\ v(\mathbf{p}, [I]; \rho) &:= (\mathbf{p}, [I]; u(\mathbf{p}, [I])\rho) \end{aligned}$$

are local trivialisations. By Proposition 3.23, these are compatible with the tame Fréchet space structures on the fibres. By Proposition 3.24, they form an atlas. •

Given the above construction, the following is an immediate consequence of Proposition 3.21.

Corollary 3.26. *The universal Dirac operator $D: E \rightarrow F$ and the universal residue map $\text{res}: E \rightarrow G$ are maps of tame Fréchet bundles.* ■

Remark 3.27. Here are some observations regarding the above discussion:

- (1) The construction of the tame Fréchet space bundle structures is quite robust, insofar as it depends only on [Proposition 3.21](#), [Proposition 3.23](#), and [Proposition 3.24](#). In particular, the method shall be used later in this article to construct certain tame Fréchet subbundles of E .
- (2) The construction of F could have been carried out with a less delicate definition of slice, because $H_b^\infty \Gamma(X \setminus Z, S \otimes I; \mathbf{p})$ does not depend on the normal structure.
- (3) For $k \in \mathbf{N}_0$ the right actions of $\text{Stab}_{\widehat{\mathcal{G}}(S)}(\text{Sl})$ on the Hilbert spaces $H_a^k(X \setminus Z_0, S \otimes I_0; \mathbf{p}_0)$, $H_b^k(X \setminus Z_0, S \otimes I_0; \mathbf{p}_0)$, and $H^k(Z_0, \check{S}_{\mathbf{p}_0})$ fail to be smooth (for the same reason that the translation action $\mathbb{R} \cup H^1(\mathbb{R})$ fails to be smooth). Therefore, E , F , and G are not (in an obvious way) inverse limits of Hilbert space bundles—at least not in the usual sense, but see [\[TW21, §4.4\]](#).
- (4) By construction, E , F , and G are $\widehat{\mathcal{G}}(S)$ -equivariant tame Fréchet space bundles over $\text{Dir} \times \text{Ram}$, and D and res are $\widehat{\mathcal{G}}(S)$ -equivariant (up to the caveat pointed out in [Footnote 1](#)). ♣

3.5 Covariant derivatives of the universal Dirac operator

This section explains how to differentiate the universal Dirac operator D . Of course, this requires the choice of (partial) connections.

Definition 3.28. The slice distribution $\mathcal{F}^{\text{Sl}} \subseteq T(\text{Dir} \times \text{Ram})$ is the distribution induced by the slices foliating $\text{Dir} \times \text{Ram}$; that is: for every $(\mathbf{p}, [I]) \in \text{Dir} \times \text{Ram}$

$$\mathcal{F}_{\mathbf{p}, [I]}^{\text{Sl}} := T_{\mathbf{p}, [I]} \text{Sl}_{\mathbf{p}, [I]}$$

with $\text{Sl}_{\mathbf{p}, [I]}$ denoting the unique slice through $(\mathbf{p}, [I])$. The canonical trivialisations of $E|_{\text{Sl}}$, $F|_{\text{Sl}}$, and $G|_{\text{Sl}}$ defined by [Proposition 3.21](#) define the slice partial covariant derivatives ∇^{Sl} with respect to $\mathcal{F}^{\text{Sl}}$ on E , F , and G . •

To compute $\nabla^{\text{Sl}} D$, it is helpful to decompose $\mathcal{F}^{\text{Sl}}$ according to [Proposition 3.2](#).

Proposition 3.29. *For every $(Z_0, [I_0]) \in \text{Ram}$, normal data (N, h, I) for Z_0 , and $(\mathbf{p}, [I]) \in \text{Sl}$ the tangent space $T_{\mathbf{p}, [I]} \text{Sl}$ decomposes into three tame Fréchet subspaces*

$$T_{\mathbf{p}, [I]} \text{Sl} = H_{\text{BG}; \mathbf{p}, [I]}^{\text{Sl}} \oplus V_{\mathcal{G}; \mathbf{p}, [I]}^{\text{Sl}} \oplus V_{\nabla, \text{Uh}; \mathbf{p}, [I]}^{\text{Sl}}$$

defined by

$$H_{\text{BG}; \mathbf{p}, [I]}^{\text{Sl}} := \{(\dot{g}; \dot{\gamma}^{\text{BG}}, \dot{\nabla}^{\text{BG}}; 0) \in H_{\text{BG}; \mathbf{p}, [I]} : \dot{g}|_Z \in \Gamma(Z, \text{Hom}(S^2 TZ, \mathbb{R}))\},$$

$$V_{\mathcal{G}; \mathbf{p}, [I]}^{\text{Sl}} := \{(0; [\xi, \gamma], -\nabla \xi; 0) \in V_{\mathcal{G}; \mathbf{p}, [I]} : [\xi|_Z, \gamma|_{\Lambda^2 N}] = 0\}, \quad \text{and}$$

$$V_{\nabla, \text{Uh}; \mathbf{p}, [I]}^{\text{Sl}} := V_{\nabla, \text{Uh}; \mathbf{p}, [I]};$$

in particular, $V_{\nabla, \mathbf{p}} \oplus 0 \subseteq T_{\mathbf{p}, [I]} \text{Sl}$.

Proof. Direct inspection reveals that $(\dot{g}; \dot{\gamma}, \dot{\nabla}; 0) \in T_{\mathbf{p}}\mathbf{Dir} \oplus T_{[I]}\mathbf{Ram}$ lies in $\mathcal{F}^{\text{Sl}}$ if and only if:

- (1) $\dot{g}|_Z \in \Gamma(Z, \text{Hom}(S^2TZ, \mathbb{R}))$ and
- (2) for every $x \in Z_0$ and every orthonormal basis (e_1, e_2) of N_x

$$\dot{\gamma}(e_1)\gamma(e_2) + \gamma(e_1)\dot{\gamma}(e_2) = 0.$$

This together with [Proposition 3.2](#) implies the assertion. ■

Proposition 3.30. *For every $(\mathbf{p}, [I]) \in \mathbf{Dir} \times \mathbf{Ram}$ the following hold:*

- (1) For every $\dot{\mathbf{p}}_{\text{BG}} = (\dot{g}; \dot{\gamma}^{\text{BG}}, \dot{\nabla}^{\text{BG}}; 0) \in H_{\text{BG}; \mathbf{p}, [I]}^{\text{Sl}}$

$$\nabla_{\dot{\mathbf{p}}_{\text{BG}}}^{\text{Sl}} \mathbf{D}(\mathbf{p}, \mathbf{I}; \phi) = -\frac{1}{2} \sum_{i,j=1}^d \dot{g}(e_i, e_j) \gamma(e_i) \nabla_{e_j} \phi + \frac{1}{4} \gamma((\nabla^* \dot{g})^\# + \nabla(\text{tr}_g \dot{g})) \phi.$$

- (2) For every $\dot{\mathbf{p}}_{\mathcal{G}} = (0; [\xi, \gamma], -\nabla \xi; 0) \in V_{\mathcal{G}; \mathbf{p}, [I]}^{\text{Sl}}$

$$\nabla_{\dot{\mathbf{p}}_{\mathcal{G}}}^{\text{Sl}} \mathbf{D}(\mathbf{p}, \mathbf{I}; \phi) = \sum_{i=1}^d ([\xi, \gamma(e_i)] \nabla_{e_i} \phi - \gamma(e_i) (\nabla_{e_i} \xi) \phi).$$

- (3) For every $\dot{\mathbf{p}}_{\nabla} = (0; 0, \dot{\nabla}; 0) \in V_{\nabla; \mathbf{p}, [I]}^{\text{Sl}}$

$$\nabla_{\dot{\mathbf{p}}_{\nabla}}^{\text{Sl}} \mathbf{D}(\mathbf{p}, \mathbf{I}; \phi) = \sum_{i=1}^d \gamma(e_i) \dot{\nabla}_{e_i} \phi.$$

Moreover, $\nabla^{\text{Sl}} \text{res} = 0$.

Proof. The variation $\dot{\#}$ of the isomorphism $\#: T^*X \rightarrow TX$ induced by $\dot{g} \in T_g \mathbf{Met}$ satisfies

$$\dot{\#} \alpha = - \sum_{i=1}^d \dot{g}(\alpha^\#, e_i) e_i.$$

Therefore, for every $(\mathbf{p}; [I]) \in \mathbf{Dir} \times \mathbf{Ram}$ and $\dot{\mathbf{p}} = (\dot{g}; \dot{\gamma}, \dot{\nabla}; 0) \in \mathcal{F}_{\mathbf{p}, [I]}^{\text{Sl}}$,

$$(3.31) \quad \nabla_{\dot{\mathbf{p}}}^{\text{Sl}} \mathbf{D}(\mathbf{p}, \mathbf{I}; \phi) = \sum_{i=1}^d (\dot{\gamma}(e_i) \nabla_{e_i} \phi + \gamma(e_i) \dot{\nabla}_{e_i} \phi) - \sum_{i,j=1}^d \dot{g}(e_i, e_j) \gamma(e_i) \nabla_{e_j} \phi.$$

For every $\dot{\mathbf{p}} = \dot{\mathbf{p}}_{\text{BG}} = (\dot{g}; \dot{\gamma}^{\text{BG}}, \dot{\nabla}^{\text{BG}}; 0)$ as in [Proposition 3.2 \(1\)](#) this concretises to

$$\begin{aligned} \nabla_{\dot{\mathbf{p}}_{\text{BG}}}^{\text{Sl}} \mathbf{D}(\mathbf{p}, \mathbf{I}; \phi) &= -\frac{1}{2} \sum_{i,j=1}^d \dot{g}(e_i, e_j) \gamma(e_i) \nabla_{e_j} \phi + \frac{1}{8} \sum_{i,j,k=1}^d (\nabla_{e_i} \dot{g})(e_j, e_k) \gamma(e_k) [\gamma(e_i), \gamma(e_j)] \phi \\ &= -\frac{1}{2} \sum_{i,j=1}^d \dot{g}(e_i, e_j) \gamma(e_i) \nabla_{e_j} \phi + \frac{1}{4} \gamma((\nabla^* \dot{g})^\# + \nabla(\text{tr}_g \dot{g})) \phi. \end{aligned}$$

The final step uses the observation

$$\gamma(e_k)[\gamma(e_i), \gamma(e_j)] + \gamma(e_j)[\gamma(e_i), \gamma(e_k)] = -2\delta_{ij}\gamma(e_k) + 4\delta_{jk}\gamma(e_i) - 2\delta_{ki}\gamma(e_j).$$

This proves (1). The remaining formulae (2) and (3) are obvious. $\blacksquare$

Remark 3.32. Direct inspection of the above formulae and a glance at [Proposition 3.30](#) reveal that

$$\begin{aligned} \nabla_{\mathbf{p}_{\text{BG}}}^{\text{Sl}} \mathbf{D}(\mathbf{p}, \mathbf{I}; -), \nabla_{\mathbf{p}_{\text{v}}}^{\text{Sl}} \mathbf{D}(\mathbf{p}, \mathbf{I}; -) &\in \text{DiffOp}_b^1(S \otimes \mathbf{I}) \quad \text{and} \\ \nabla_{\mathbf{p}_{\mathcal{E}}}^{\text{Sl}} \mathbf{D}(\mathbf{p}, \mathbf{I}; -) &\in \text{DiffOp}_b^0(S \otimes \mathbf{I}) D_{\mathbf{p}}^{\mathbf{I}} + \text{DiffOp}_b^1(S \otimes \mathbf{I}). \end{aligned} \quad \clubsuit$$

Remark 3.33. The differential operator $\nabla_{\mathbf{p}_{\text{BG}}}^{\text{Sl}} \mathbf{D}(\mathbf{p}, \mathbf{I}; -)$ is not formally self-adjoint. In fact, a brief computation shows that its formal adjoint satisfies

$$\nabla_{\mathbf{p}_{\text{BG}}}^{\text{Sl}} \mathbf{D}(\mathbf{p}, \mathbf{I}; -)^* = \nabla_{\mathbf{p}_{\text{BG}}}^{\text{Sl}} \mathbf{D}(\mathbf{p}, \mathbf{I}; -) - \frac{1}{2} \gamma(\nabla(\text{tr}_g \dot{g})).$$

This can be rectified by using the **volume renormalized slice partial covariant derivative**

$$\nabla_{\mathbf{p}}^{\text{Sl,vol}} := \nabla_{\mathbf{p}}^{\text{Sl}} + \frac{1}{4} \text{tr}_g \dot{g}$$

instead of ∇^{Sl} ; cf. [\[Mai97, §2.2; AD25, §2.5\]](#). For the purposes of this article, the above is not an issue and it suffices to work with ∇^{Sl} . However, it would be one in setting up a Brill–Noether theory of $\mathbb{Z}/2\mathbb{Z}$ harmonic spinors; cf. [\[DW23, §1.A\]](#). $\clubsuit$

The right action of $\widehat{\mathcal{G}}(S)$ on $\mathbf{Dir} \times \mathbf{Ram}$ induces a right action

$$T(\mathbf{Dir} \times \mathbf{Ram}) \cup T\widehat{\mathcal{G}}(S) = \widehat{\mathcal{G}}(S) \ltimes \text{Lie } \widehat{\mathcal{G}}(S).$$

This can be used to construct (local) partial covariant derivatives with respect to

$$\mathcal{F}^{\text{Ram}} := \text{pr}_2^* T\mathbf{Ram} \subseteq T(\mathbf{Dir} \times \mathbf{Ram})$$

from ∇^{Sl} and the following infinitesimal version of [Proposition 3.22](#).

Proposition 3.34. *For every $(\mathbf{p}_0, [\mathbf{I}_0]) \in \mathbf{Dir} \times \mathbf{Ram}$ there is a **local infinitesimal uniformiser**; that is: an open neighbourhood $(\mathbf{p}_0, [\mathbf{I}_0]) \in \mathcal{U} \subseteq \mathbf{Dir} \times \mathbf{Ram}$ and a tame smooth map*

$$\begin{aligned} \mathcal{F}^{\text{Ram}}|_{\mathcal{U}} &\rightarrow \text{Vect}(X) \\ (\mathbf{p}, [\mathbf{I}]; v) &\mapsto \bar{v} = \bar{v}(\mathbf{p}, [\mathbf{I}]; v) \end{aligned}$$

such that:

(1) for every $(\mathbf{p}, [\mathbf{I}]; v) \in \mathcal{F}^{\text{Ram}}|_{\mathcal{U}}$

$$(\mathbf{p}, [\mathbf{I}]; v) \cdot \text{lift}_{\mathbf{p}}^K(\bar{v}) \in \mathcal{F}_{\mathbf{p}, [\mathbf{I}]}^{\text{Sl}},$$

(2) $\bar{v}(\mathbf{p}, [\mathbf{I}]; -)$ is $\mathbb{R}$ -linear, and

(3) for every $(\mathbf{p}, [\mathbf{I}]; v) \in \mathcal{F}^{\text{Ram}}|_{\mathcal{U}}$

$$\|\bar{v}(\mathbf{p}, [\mathbf{I}]; v)\|_{H^{k+1}} \lesssim_k \|v\|_{H^k}.$$

Moreover, given a compact subset $K \subseteq X \setminus \text{Br}(\mathbf{I}_0)$, the local infinitesimal uniformiser can be chosen such that $\bar{v}(\mathbf{p}, [\mathbf{I}]; v)|_K = 0$ for every $(\mathbf{p}, [\mathbf{I}]; v) \in \mathcal{F}^{\text{Ram}}|_{\mathcal{U}}$.

The proof of [Proposition 3.34 \(3\)](#) uses the following well-known observation.

Lemma 3.35. *Let $d, k \in \mathbb{N}_0$. There is a 1-jet extension map*

$$\text{ext}^{(1)}: \mathcal{S}(\mathbb{R}^d) \oplus \mathcal{S}(\mathbb{R}^d)^k \rightarrow \mathcal{S}(\mathbb{R}^d \oplus \mathbb{R}^k)$$

such that for every $(f, g_1, \dots, g_k) \in \mathcal{S}(\mathbb{R}^d)^{k+1}$ its extension $F := \text{ext}^{(1)}(f, g_1, \dots, g_k) \in \mathcal{S}(\mathbb{R}^d \oplus \mathbb{R}^k)$ satisfies

$$(3.36) \quad F(-, 0) = f \quad \text{and} \quad \frac{\partial F}{\partial y_i}(-, 0) = g_i,$$

and for every $s \in \mathbb{R}$

$$(3.37) \quad \|F\|_{H^s} \lesssim_s \|f\|_{H^{s-k/2}} + \sum_{i=1}^k \|g_i\|_{H^{s-k/2-1}}.$$

Proof. Choose a $\chi \in \mathcal{S}(\mathbb{R}^k)$ with $\chi(0) = 1$ and $\nabla^\ell \chi(0) = 0$ for every $\ell \in \mathbb{N}$. For $(f, g_1, \dots, g_k) \in \mathcal{S}(\mathbb{R}^d)^{k+1}$ define $F := \text{ext}^{(1)}(f, g_1, \dots, g_k)$ by

$$\begin{aligned} F(x, y) &:= \int_{\mathbb{R}^d} e^{2\pi i \langle \xi, x \rangle} \chi(\langle \xi \rangle y) \hat{f}(\xi) \, d\xi \\ &\quad + \sum_{i=1}^k y_i \int_{\mathbb{R}^d} e^{2\pi i \langle \xi, x \rangle} \chi(\langle \xi \rangle y) \hat{g}_i(\xi) \, d\xi. \end{aligned}$$

(3.36) is an immediate consequence of the Fourier inversion theorem.

By direct computation, with the substitution $\tilde{\eta} = \eta / \langle \xi \rangle$,

$$\begin{aligned} \left\| \int_{\mathbb{R}^d} e^{2\pi i \langle \xi, x \rangle} \chi(\langle \xi \rangle y) \hat{f}(\xi) \, d\xi \right\|_{H^s}^2 &= \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 \int_{\mathbb{R}^k} \langle (\xi, \eta) \rangle^{2s} |\mathcal{F}_y(\chi(\langle \xi \rangle y))(\eta)|^2 \, d\eta \, d\xi \\ &= \int_{\mathbb{R}^d} \langle \xi \rangle^{2s-k} |\hat{f}(\xi)|^2 \underbrace{\int_{\mathbb{R}^k} \langle \tilde{\eta} \rangle^{2s} |\hat{\chi}(\tilde{\eta})|^2 \, d\tilde{\eta}}_{=: c(s)} \, d\xi \\ &= c(s) \|f\|_{H^{s-k/2}}^2, \end{aligned}$$

and

$$\begin{aligned} \left\| y_i \int_{\mathbb{R}^d} e^{2\pi i \langle \xi, x \rangle} \chi(\langle \xi \rangle y) \hat{g}_i(\xi) \, d\xi \right\|_{H^s}^2 &= \int_{\mathbb{R}^d} |\hat{g}_i(\xi)|^2 \int_{\mathbb{R}^k} \langle (\xi, \eta) \rangle^{2s} |\mathcal{F}_y(y_i \chi(\langle \xi \rangle y))(\eta)|^2 \, d\eta \, d\xi \\ &= \int_{\mathbb{R}^d} \langle \xi \rangle^{2s-k-2} |\hat{g}_i(\xi)|^2 \underbrace{\int_{\mathbb{R}^k} \langle \eta \rangle^{2s} |\mathcal{F}_y(y_i \chi(y))(\eta)|^2 \, d\eta}_{=: c_i(s)} \, d\xi \\ &= c_i(s) \|g_i\|_{H^{s-k/2-1}}^2. \end{aligned}$$

Here $\mathcal{F}_y$ denotes the Fourier transform in the variable $y \in \mathbb{R}^k$. Since $c(s), c_i(s) < \infty$, the above prove (3.37). $\blacksquare$

Proof of Proposition 3.34. Let $(\mathbf{p}; Z, [\mathbf{I}]; v) \in \mathcal{F}^{\text{Ram}}|_{\mathcal{U}}$ and $\bar{v} \in \text{Vect}(X)$. By Proposition 3.15 (2),

$$(\mathbf{p}, [\mathbf{I}]; v) \cdot \text{lift}_{\mathbf{p}}^K(\bar{v}) = (\mathcal{L}_{\bar{v}}g; \dot{\gamma}^{\text{BG}}, \dot{\nabla}^{\text{BG}} + F_{\nabla}^{\text{tw}}(\bar{v}, -); v - \bar{v}|_Z^N)$$

with $(\mathcal{L}_{\bar{v}}g; \dot{\gamma}^{\text{BG}}, \dot{\nabla}^{\text{BG}}) \in T_{\mathbf{p}}\mathbf{Dir}$ denoting the Bourguignon–Gauduchon lift of $\mathcal{L}_{\bar{v}}g \in T_g\mathbf{Met}$. Here $-^N$ denotes the orthogonal projection onto NZ . By inspection of Proposition 3.29, condition (1) translates into

$$\bar{v}|_Z^N = v \quad \text{and} \quad (\mathcal{L}_{\bar{v}}g)|_Z \in \Gamma(Z, \text{Hom}(S^2TZ, \mathbb{R})).$$

By (3.18), the second condition means that for every $x \in Z$, $t \in T_xZ$, $n, m \in N_xZ$

$$g(\nabla_n \bar{v}, t) = -g(n, \nabla_t \bar{v}) \quad \text{and} \quad g(\nabla_n \bar{v}, m) + g(n, \nabla_m \bar{v}) = 0.$$

These are algebraic conditions on the 1-jet $J^1\bar{v}|_Z$ of $\bar{v}$ at Z . Evidently, it is possible to algebraically solve for $J^1\bar{v} \in \Gamma(Z, J^1TX)$ and to extend it to a vector field $\bar{v} \in \text{Vect}(X)$. In fact, using Lemma 3.35, this can be done locally in $\mathbf{Dir} \times \mathbf{Ram}$ in such a way that $\bar{v}$ depends tame smoothly on $(\mathbf{p}, [\mathbf{I}]; v)$ and satisfies (2) and (3). $\blacksquare$

Definition 3.38. Given a local infinitesimal uniformiser $\mathcal{F}^{\text{Ram}}|_{\mathcal{U}} \rightarrow \text{Vect}(X)$, $(\mathbf{p}, [\mathbf{I}]; v) \mapsto \bar{v} = \bar{v}(\mathbf{p}, [\mathbf{I}], v)$ as in Proposition 3.34, define the partial covariant derivatives ∇^{Ram} with respect to $\mathcal{F}^{\text{Ram}}$ on $\mathbf{E}|_{\mathcal{U}}$, $\mathbf{F}|_{\mathcal{U}}$, and $\mathbf{G}|_{\mathcal{U}}$ by

$$\nabla_v^{\text{Ram}} := \nabla_{(\mathbf{p}, [\mathbf{I}], v) \cdot \text{lift}_{\mathbf{p}}^K(\bar{v})}^{\text{Sl}}. \quad \bullet$$

Unsurprisingly, $\nabla_v^{\text{Ram}}\mathbf{D}$ can be expressed in terms of the Kosmann Lie derivative defined in Proposition 3.15. This is crucial for the discussion in Section 4.

Proposition 3.39 (cf. [Kos72, §V.1.2]). *In the situation of Definition 3.38, for every $(\mathbf{p}; [\mathbf{I}]; \phi) \in \mathbf{E}|_{\mathcal{U}}$ and $v \in T_{[\mathbf{I}]}^{\text{Ram}}$*

$$\nabla_v^{\text{Ram}}\mathbf{D}(\mathbf{p}, [\mathbf{I}]; \phi) = [\mathcal{L}_{\bar{v}}^K, D_{\mathbf{p}}^{\mathbf{I}}]\phi.$$

Proof. By Proposition 3.15,

$$\begin{aligned} [\mathcal{L}_{\bar{v}}^K, \gamma(e_i)] &= \dot{\gamma}^{\text{BG}}(e_i) + \gamma([\bar{v}, e_i]) \quad \text{and} \\ [\mathcal{L}_{\bar{v}}^K, \nabla_{e_i}] &= \dot{\nabla}_{e_i}^{\text{BG}} + F_{\nabla}^{\text{tw}}(\bar{v}, e_i) + \nabla_{[\bar{v}, e_i]}. \end{aligned}$$

Therefore,

$$[\mathcal{L}_{\bar{v}}^K, D] = \sum_{i=1}^d \dot{\gamma}^{\text{BG}}(e_i) \nabla_{e_i} + \gamma(e_i) \dot{\nabla}_{e_i}^{\text{BG}} + \gamma(e_i) F_{\nabla}^{\text{tw}}(\bar{v}, e_i) - \gamma(\nabla_{e_i} \bar{v}) \nabla_{e_i} - \gamma(e_i) \nabla_{\nabla_{e_i} \bar{v}}.$$

A glance at (3.31) reveals that this is precisely $\nabla_v^{\text{Ram}}\mathbf{D}(\mathbf{p}, [\mathbf{I}]; -)$. $\blacksquare$

Remark 3.40. The formula in Proposition 3.39 has to be treated with some care, because

$$\mathcal{L}_{\bar{v}}^K D_{\mathbf{p}}^{\mathbf{I}} \quad \text{and} \quad D_{\mathbf{p}}^{\mathbf{I}} \mathcal{L}_{\bar{v}}^K$$

do not map $\mathbf{E}_{\mathbf{p}, \mathbf{I}}$ to $\mathbf{F}_{\mathbf{p}, \mathbf{I}}$, but their difference does. In fact, as a consequence of Remark 3.32,

$$\nabla_v^{\text{Ram}}\mathbf{D}(\mathbf{p}, \mathbf{I}; -) \in \text{DiffOp}_b^1(S \otimes \mathbf{I}). \quad \clubsuit$$

3.6 Families of residue conditions

This subsection explains how to impose residue conditions in families.

Definition 3.41. A family of residue conditions $(B, f, \mathbf{R})$ consists of:

- (1) a tame Fréchet manifold B ,
- (2) a tame smooth map $f: B \rightarrow \mathbf{Dir} \times \mathbf{Ram}$, and
- (3) a tame Fréchet space subbundle $\mathbf{R} \subseteq f^*\mathbf{G}$. •

Here is an example of a family of (spectral) residue conditions.

Example 3.42. Let $\lambda \in \mathbb{R}$. Denote by $\mathcal{U}_\lambda \subseteq \mathbf{Dir} \times \mathbf{Ram}$ the open subset of those $(\mathbf{p}, [\mathbf{I}]) \in \mathbf{Dir} \times \mathbf{Ram}$ for which the branching locus operator $A_{\mathbf{p}, [\mathbf{I}]}$ does not have λ as an eigenvalue; that is: $\lambda \notin \text{spec}(A_{\mathbf{p}, [\mathbf{I}]})$. There is a tame Fréchet space subbundle

$$\mathbf{APS}_\lambda \subseteq \mathbf{G}|_{\mathcal{U}_\lambda}$$

such that, for every $(\mathbf{p}, [\mathbf{I}]) \in \mathcal{U}_\lambda$,

$$\mathbf{APS}_{\lambda; \mathbf{p}, [\mathbf{I}]} = \mathbf{1}_{(-\infty, \lambda)}(A_{\mathbf{p}, [\mathbf{I}]})\Gamma(\text{Br}(\mathbf{I}), \check{S}_{\mathbf{p}}).$$

This defines the family of APS residue conditions with threshold $\lambda \in \mathbb{R}$. ♠

Proposition 3.43. For every family of residue conditions $(B, f, \mathbf{R})$

$$\mathbf{E}_\mathbf{R} := (f^*\text{res})^{-1}(\mathbf{R}) \subseteq f^*\mathbf{E}$$

is a tame Fréchet space subbundle.

$$\begin{array}{ccccc} \mathbf{E}_\mathbf{R} & \xrightarrow{\mathbf{D}_\mathbf{R}} & f^*\mathbf{F} & & \\ \downarrow \text{res} & \searrow & \downarrow & \swarrow & \\ \mathbf{R} & & f^*\mathbf{E} & \xrightarrow{f^*\mathbf{D}} & f^*\mathbf{F} \\ & \searrow & \downarrow f^*\text{res} & & \\ & & f^*\mathbf{G} & & \end{array}$$

Figure 2: The construction of $\mathbf{D}_\mathbf{R}$.

Definition 3.44. In the situation of [Proposition 3.43](#), the restriction of $f^*\mathbf{D}$ to $\mathbf{E}_\mathbf{R}$ is denoted by

$$\mathbf{D}_\mathbf{R}: \mathbf{E}_\mathbf{R} \rightarrow f^*\mathbf{F}. \quad \bullet$$

The proof of [Proposition 3.43](#) relies on the following.

Proposition 3.45. *The **universal bundle of $\mathbb{Z}/2\mathbb{Z}$ spinors***

$$E_0 := \ker \text{res} \subseteq E$$

is a tame Fréchet space subbundle. Moreover, the exact sequence

$$E_0 \hookrightarrow E \xrightarrow{\text{res}} G$$

of tame Fréchet space bundles locally splits; that is: for every $(p, [I]) \in \text{Dir} \times \text{Ram}$ there are an open neighbourhood $(p, [I]) \in \mathcal{U} \subseteq \text{Dir} \times \text{Ram}$ and a map of tame Fréchet space bundles $\text{ext}: G|_{\mathcal{U}} \rightarrow E|_{\mathcal{U}}$ satisfying $\text{res} \circ \text{ext} = 1$.

Proof. That $E_0 \subseteq E$ is a tame Fréchet space subbundle is evident by inspection of the trivialisations constructed in [Definition 3.25](#). The splitting arises from the extension map mentioned in [Section 2.2](#) and constructed in [\[BW26, Definition 3.40 \(2\)\]](#). ■

Remark 3.46. Here are some observations regarding E_0 :

- (1) The construction of E_0 could be carried out directly with a less delicate definition of slice, because $\ker \text{res} = H_a^\infty \Gamma(X \setminus Z, S \otimes I; 0; p)$ does not depend on the normal structure; cf. [Remark 3.27 \(2\)](#).
- (2) For every $v \in \text{Vect}(X)$ and $(p, [I]; \Phi) \in E_0$, $(p, [I]; \nabla_v \Phi) \in F$ and the corresponding map

$$\nabla: \text{Vect}(X) \times E_0 \rightarrow F$$

is tame smooth [\[BW26, Proposition 4.27\]](#). ♣

Proof of [Proposition 3.43](#). Using [Proposition 3.45](#), E_R can locally be presented as

$$E_R|_{f^{-1}(\mathcal{U})} = f^* E_0 \oplus (f^* \text{ext})(R). \quad \blacksquare$$

The arguments in [Section 4](#) use families of residue conditions of the following kind.

Definition 3.47. Here is how to consider local residue conditions in families:

- (1) (a) The **universal branching locus** is the tame Fréchet submanifold

$$\text{Br} := \{(p, [I]; x) \in \text{Dir} \times \text{Ram} \times X : x \in \text{Br}(I)\} \subseteq \text{Dir} \times \text{Ram} \times X.$$

The projection map $p: \text{Br} \rightarrow \text{Dir} \times \text{Ram}$ is a fibre bundle.

- (b) The normal bundle $N := N\text{Br}$ shall be identified with

$$N = \bigsqcup_{(p, [I]) \in \text{Dir} \times \text{Ram}} N\text{Br}(I).$$

It inherits a Euclidean metric from the metrics on X which are part of $p \in \text{Dir}$. This induces an isomorphism $(\Lambda^2 N)^{\otimes 2} \cong \mathbb{R}$ and endows

$$\mathbb{A} := \mathbb{R} \oplus i\Lambda^2 N$$

with the structure of a bundle of algebras over Br , whose fibres are isomorphic to $\mathbb{C}$, but canonically only up to complex conjugation.

- (c) The construction from [BW26, Definition 3.21 and Remark 3.22] explained in [Section 2.2](#), parametrised by $(\mathbf{p}, [\mathbf{I}]) \in \mathbf{Dir} \times \mathbf{Ram}$, assembles into an $\mathbb{A}$ -module $N^{-1/2}$.
- (d) Clifford multiplication endows the restriction of $\mathrm{pr}_X^* S$ to $\mathbf{Br}$ with the structure of an $\mathbb{A}$ -module. The **universal residue bundle** is the rank r Euclidean bundle over $\mathbf{Br}$ defined by

$$\check{S} := \mathrm{pr}_X^* S \otimes_{\mathbb{A}} N^{-1/2} = \bigsqcup_{(\mathbf{p}, [\mathbf{I}]) \in \mathbf{Dir} \times \mathbf{Ram}} \check{S}_{\mathbf{p}}.$$

- (2) A family of local residue conditions $(\mathbf{B}, f, \mathbf{V})$ consists of:

- (a) a tame Fréchet manifold $\mathbf{B}$,
- (b) a tame smooth map $f: \mathbf{B} \rightarrow \mathbf{Dir} \times \mathbf{Ram}$, and
- (c) a subbundle $\mathbf{V} \subseteq (p^* f)^* \check{S}$.

Here $f^* p: f^* \mathbf{Br} \rightarrow \mathbf{B}$ is the pullback of the universal branching locus and $p^* f: f^* \mathbf{Br} \rightarrow \mathbf{Br}$ is the projection map; see [Figure 3](#).

- (3) A family of local residue conditions $(\mathbf{B}, f, \mathbf{V})$ defines a family of residue conditions $(\mathbf{B}, f, \mathbf{R})$ such that for every $b \in \mathbf{B}$

$$\mathbf{R}_b := \Gamma(Z_b, \mathbf{V}|_{Z_b}) \subseteq \Gamma(Z_b, \check{S}_{\mathbf{p}_b}) \quad \text{with} \quad (\mathbf{p}_b; Z_b, [\mathbf{I}_b]) := f(b). \quad \bullet$$

Remark 3.48. Here are some comments regarding [Definition 3.47](#):

- (1) $\check{S}$ is related to the universal bundle of residues $\mathbf{G}$, constructed in [Section 3.4](#), by

$$\mathbf{G} = p_* \check{S};$$

in the sense that, for every $(\mathbf{p}, [\mathbf{I}]) \in \mathbf{Dir} \times \mathbf{Ram}$, $\mathbf{G}_{\mathbf{p}, [\mathbf{I}]} = \Gamma(p^{-1}(\mathbf{p}, [\mathbf{I}]), \check{S})$.

- (2) Identifying $f^* \mathbf{G} = f^* p_* \check{S} = (f^* p)_* (p^* f)^* \check{S}$, the tame Fréchet space subbundle $\mathbf{R} \subseteq f^* \mathbf{G}$ is

$$\mathbf{R} := (f^* p)_* \mathbf{V} \subseteq f^* \mathbf{G}. \quad \clubsuit$$

For the purposes of this article, it is very desirable for $\mathbf{D}_{\mathbf{R}}$ to have the following property.

Definition 3.49. Let E and F be tame Fréchet space bundles over a tame Fréchet manifold B . A map of tame Fréchet space bundles $L: E \rightarrow F$ is **uniformly Fredholm** if for every $b \in B$ there exist an open neighbourhood $b \in \mathcal{U} \subseteq B$, finite-rank vector bundles Def and Ob over $\mathcal{U}$, and maps of tame Fréchet space bundles $\pi: E|_{\mathcal{U}} \twoheadrightarrow \mathrm{Def}$ and $\iota: \mathrm{Ob} \hookrightarrow F|_{\mathcal{U}}$ such that the **thickening**

$$\begin{pmatrix} L & \iota \\ \pi & 0 \end{pmatrix}: E|_{\mathcal{U}} \oplus \mathrm{Ob} \rightarrow F|_{\mathcal{U}} \oplus \mathrm{Def}$$

is an isomorphism of tame Fréchet space bundles. •

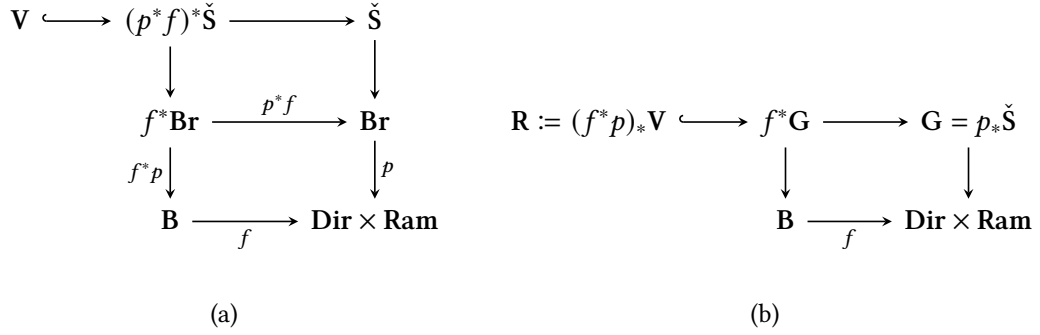


Figure 3: Families of local residue conditions.

Proposition 3.50. *Let $(\mathbf{B}, f, \mathbf{V})$ be a family of local residue conditions such that for every $b \in \mathbf{B}$, $x \in Z_b$, and $v \in T_x Z_b \setminus \{0\}$*

$$\mathbf{V}_{(b,x)} \oplus J_{\mathbf{p}_b} \gamma(v) \mathbf{V}_{(b,x)} = \check{\mathbf{S}}_{\mathbf{p}_b, x}.$$

Denote by $(\mathbf{B}, f, \mathbf{R})$ the corresponding family of residue conditions. If $\alpha: f^\mathbf{F} \rightarrow f^*\mathbf{F}$ is a map of tame Fréchet bundles of order zero, then*

$$\mathbf{D}_{\mathbf{R}} + \alpha: \mathbf{E}_{\mathbf{R}} \rightarrow f^*\mathbf{F}$$

is uniformly Fredholm. Moreover, for every $b_0 \in \mathbf{B}$ the maps $\pi: \mathbf{E}_{\mathbf{R}}|_{\mathcal{U}} \rightarrow \mathbf{Def}$ and $\iota: \mathbf{Ob} \hookrightarrow f^\mathbf{F}|_{\mathcal{U}}$ in [Definition 3.49](#) can be chosen such that there is a compact subset $K \subseteq X$ such that for every $b \in \mathcal{U}$ the following hold:*

- (1) $K \subseteq X \setminus Z_b$;
- (2) if $\phi \in \mathbf{E}_{\mathbf{R},b}$ has $\text{supp}(\phi) \cap K = \emptyset$, then $\pi(\phi) = 0$; and
- (3) $\text{supp}(\iota(o)) \subseteq K$ for every $o \in \mathbf{Ob}_b$.

Proof. By [\[BW26, Proposition 4.41\]](#), for every $b \in \mathbf{B}$ the residue condition $\mathbf{R}_b$ is elliptic; in particular: for every $k \in \mathbf{N}_0$ the operator

$$D_{\mathbf{R}_b}^{(k)} + \mathbf{a}_b^{(k)} := D_{\mathbf{p}_b, \mathbf{R}_b}^{I_b, (k)} + \mathbf{a}_b^{(k)}: H_a^{k+1} \Gamma(X \setminus Z_b, S \otimes I_b; \mathbf{R}_b; \mathbf{p}_b) \rightarrow H_b^k \Gamma(X \setminus Z_b, S \otimes I_b; \mathbf{p}_b)$$

is Fredholm, and $\ker(D_{\mathbf{R}_b}^{(k)} + \mathbf{a}_b^{(k)})$ and $\text{coker}(D_{\mathbf{R}_b}^{(k)} + \mathbf{a}_b^{(k)})$ are independent of $k \in \mathbf{N}_0$.

Let $b_0 \in \mathbf{B}$. Choose the following:

- (I) a compact subset $K \subseteq X \setminus Z_{b_0}$ with non-empty interior,
- (II) an open neighbourhood $b_0 \in \mathcal{U} \subseteq \mathbf{B}$ such that $K \subseteq X \setminus Z_b$ for every $b \in \mathcal{U}$,
- (III) finite rank Euclidean vector bundles $\mathbf{Def}$ and $\mathbf{Ob}$ over $\mathcal{U}$ and maps of tame Fréchet space bundles $\pi: \mathbf{E}_{\mathbf{R}}|_{\mathcal{U}} \rightarrow \mathbf{Def}$ and $\iota: \mathbf{Ob} \hookrightarrow f^*\mathbf{F}|_{\mathcal{U}}$ such that for every $b \in \mathcal{U}$:

(a) for every $\phi \in \mathbf{E}_{\mathbf{R},b}$

$$\|\pi(\phi)\| \lesssim \|\phi\|_{H_a^1},$$

(b) (i) if $\text{supp}(\phi) \cap K = \emptyset$, then $\pi(\phi) = 0$,

(ii) $\text{supp}(\iota(o)) \subseteq K$ for every $b \in \mathcal{U}$ and $o \in \text{Ob}_b$,

(c) for every $b \in \mathcal{U}$ and $k \in \mathbf{N}_0$

$$L_b^{(k)} := \begin{pmatrix} D_{\mathbf{R}_b}^{(k)} + \mathbf{a}_b^{(k)} & \iota_b \\ \pi_b & 0 \end{pmatrix}$$

is invertible.

(III.b) can be arranged with the help of [Proposition 3.22](#). To see that (III.c) can be achieved, observe the following. If $L_b^{(0)}$ is invertible, then $L_b^{(k)}$ is invertible for every $k \in \mathbf{N}_0$ by elliptic regularity [[BW26](#), Theorem 4.17]. Since $b \mapsto L_b^{(0)}$ is a continuous family of bounded operators between Banach spaces (in a suitable trivialisation), it suffices to ensure that $L_{b_0}^{(0)}$ is invertible. The latter holds provided the induced maps $\ker(D_{\mathbf{R}_{b_0}}^{(0)} + \mathbf{a}_{b_0}^{(0)}) \rightarrow \text{Def}_{b_0}$ and $\text{Ob}_{b_0} \rightarrow \text{coker}(D_{\mathbf{R}_{b_0}}^{(0)} + \mathbf{a}_{b_0}^{(0)})$ are isomorphisms. Because the kernels of both $D_{\mathbf{R}_{b_0}} + \mathbf{a}_{b_0}$ and its adjoint have the unique continuation property, this can easily be arranged while satisfying the desired support properties for a suitable $K \subseteq X \setminus Z_{b_0}$.

The argument in [[Ham82](#), Part II Proofs of Theorem 3.3.3 and Theorem 3.3.5] shows that inverses obtained from (III.c) assemble into an inverse of the map of tame Fréchet bundles

$$\begin{pmatrix} D_{\mathbf{R}} + \mathbf{a} & \iota \\ \pi & 0 \end{pmatrix} : \mathbf{E}_{\mathbf{R}}|_{\mathcal{U}} \oplus \text{Ob} \rightarrow f^* \mathbf{F} \oplus \text{Def}. \quad \blacksquare$$

3.7 The universal subbundle of admissible $\mathbb{Z}/2\mathbb{Z}$ spinors

Naively, one might hope to use $\mathbf{E}_0$ as the ambient space for the construction of tame Fréchet manifold structures on the universal moduli spaces of $\mathbb{Z}/2\mathbb{Z}$ harmonic spinors introduced in [Section 1](#). However, for the arguments in [Section 4.3](#) it is crucial to resolve the issue raised in [Remark 3.40](#) by working with a smaller ambient tame Fréchet space bundle $\mathbf{E}_{00}$, the **universal bundle of admissible $\mathbb{Z}/2\mathbb{Z}$ spinors**. The following describes the fibres of $\mathbf{E}_{00}$.

Proposition 3.51. *For every $\mathbf{p} \in \mathbf{Dir}$ and $(Z, [I]) \in \mathbf{Ram}$ the graded Fréchet space $(K^\infty(X \setminus Z, S \otimes I; \mathbf{p}), (\|\cdot\|_{K^k})_{k \in \mathbf{N}_0})$ defined by*

$$K^\infty(X \setminus Z, S \otimes I; \mathbf{p}) := H_a^\infty(X \setminus Z, S \otimes I; 0; \mathbf{p}) \cap (D_{\mathbf{p}}^I)^{-1} w_{\log}^{-1} H_a^\infty(X \setminus Z, S \otimes I; 0; \mathbf{p}), \quad \text{and}$$

$$\|\phi\|_{K^k} := \|\phi\|_{H_b^{k+2}} + \|\nabla \phi\|_{H_b^{k+1}} + \|w_{\log} D_{\mathbf{p}}^I \phi\|_{H_b^{k+1}} + \|\nabla w_{\log} D_{\mathbf{p}}^I \phi\|_{H_b^k}$$

is tame; that is, it has the smoothing operator property. Here $w_{\log}$ denotes multiplication by $\langle \log(r) \rangle = \sqrt{1 + \log(r)^2}$ and r denotes the distance to Z .

The proof uses the following observation.

Proposition 3.52. For every $\mathbf{p} \in \text{Dir}$, $(Z, [\mathbf{I}]) \in \text{Ram}$, $\phi \in K^\infty(X \setminus Z, S \otimes \mathbf{I}; \mathbf{p})$, and $k \in \mathbb{N}_0$

$$\|w_{\log} \phi\|_{H_b^{k+2}} + \|r^{-1} w_{\log} \phi\|_{H_b^{k+1}} + \|w_{\log} \nabla \phi\|_{H_b^{k+1}} \lesssim_k \|\phi\|_{K^k}.$$

Proof. By [BW26, Lemma 3.3 and Proposition 4.27],

$$(3.53) \quad \|r^{-1} \phi\|_{H_b^{k+1}} \lesssim_k \|\phi\|_{H_b^{k+1}} + \|\nabla \phi\|_{H_b^{k+1}} \lesssim_k \|\phi\|_{K^k}.$$

Therefore, since $[D_{\mathbf{p}}^{\mathbf{I}}, \log(r)] = r^{-1} \gamma(\partial_r)$,

$$\|w_{\log} \phi\|_{H_b^{k+1}} + \|D_{\mathbf{p}}^{\mathbf{I}} w_{\log} \phi\|_{H_b^{k+1}} \lesssim_k \|\phi\|_{H_b^{k+1}} + \|\nabla \phi\|_{H_b^{k+1}} + \|w_{\log} D_{\mathbf{p}}^{\mathbf{I}} \phi\|_{H_b^{k+1}} \leq \|\phi\|_{K^k}.$$

As a consequence, $w_{\log} \phi \in H_a^{k+1}(X \setminus Z, S \otimes \mathbf{I})$. Since $\phi \in L^\infty(X \setminus Z, S \otimes \mathbf{I})$, $\text{res}(w_{\log} \phi) = 0$. By [BW26, Theorem 4.11],

$$\|w_{\log} \phi\|_{H_b^{k+2}} \lesssim_k \|w_{\log} \phi\|_{H_b^{k+1}} + \|D_{\mathbf{p}}^{\mathbf{I}} w_{\log} \phi\|_{H_b^{k+1}}$$

and $w_{\log} \phi \in H_a^{k+2}(X \setminus Z, S \otimes \mathbf{I}; 0; \mathbf{p})$. Therefore, by [BW26, Lemma 3.3 and Proposition 4.27],

$$\|r^{-1} w_{\log} \phi\|_{H_b^{k+1}} + \|w_{\log} \nabla \phi\|_{H_b^{k+1}} \lesssim_k \|r^{-1} \phi\|_{H_b^{k+1}} + \|\nabla w_{\log} \phi\|_{H_b^{k+1}} \lesssim \|\phi\|_{K^k}. \quad \blacksquare$$

Proof of Proposition 3.51. For every $\phi \in \Gamma(X \setminus Z, S \otimes \mathbf{I})$ with $\text{supp } \phi \subseteq r^{-1}([0, 1/2])$ set

$$\|\phi\|_{\dot{K}^k} := \|\phi\|_{H_b^{k+2}} + \|\dot{\nabla} \phi\|_{H_b^{k+1}} + \|\log(r) \dot{D}_{\mathbf{p}}^{\mathbf{I}} \phi\|_{H_b^{k+1}} + \|\dot{\nabla} \log(r) \dot{D}_{\mathbf{p}}^{\mathbf{I}} \phi\|_{H_b^k}.$$

Here $\dot{\nabla}$ and $\dot{D}_{\mathbf{p}}^{\mathbf{I}}$ are the models for ∇ and $D_{\mathbf{p}}^{\mathbf{I}}$ constructed in [BW26, §3.2]. Since $B := D_{\mathbf{p}}^{\mathbf{I}} - \dot{D}_{\mathbf{p}}^{\mathbf{I}} \in \text{DiffOp}_b^1(S \otimes \mathbf{I})$, $[B, \log(r)] = r^{-1} \sigma_B(\text{dr})$ and its conormal derivatives are bounded. Therefore, for $\phi \in K^\infty(X \setminus Z, S \otimes \mathbf{I}; \mathbf{p})$ with $\text{supp } \phi \subseteq r^{-1}([0, 1/2])$, by Proposition 3.52

$$\|\log(r) B \phi\|_{H_b^{k+1}} \lesssim_{B,k} \|\log(r) \phi\|_{H_b^{k+2}} \lesssim_{B,k} \|\phi\|_{K^k}$$

and

$$\begin{aligned} \|\nabla \log(r) B \phi\|_{H_b^k} &\lesssim_k \|r^{-1} B \phi\|_{H_b^k} + \|\log(r) \nabla B \phi\|_{H_b^k} \\ &\lesssim_{B,k} \|r^{-1} \phi\|_{H_b^{k+1}} + \|\log(r) \nabla \phi\|_{H_b^{k+1}} \lesssim_k \|\phi\|_{K^k}. \end{aligned}$$

The above (and similar estimates) imply that

$$\|\phi\|_{\dot{K}^k} \lesssim_k \|\phi\|_{K^k} \quad \text{and} \quad \|\phi\|_{K^k} \lesssim_k \|\phi\|_{\dot{K}^k}.$$

Appendix A constructs a family of smoothing operators for $(\|\cdot\|_{\dot{K}^k} : k \in \mathbb{N}_0)$. These combined with the usual smoothing operators away from Z construct the required smoothing operators as, for example, in [Par23, Proof of Lemma B.1; BW26, Proof of Proposition 4.32]. $\blacksquare$

Proposition 3.54. *For every $\mathbf{p} \in \text{Dir}$, $(Z, [\mathbf{I}]) \in \text{Ram}$, $\Phi \in K^\infty(X \setminus Z, S \otimes \mathbf{I}; \mathbf{p})$, and $\bar{v} \in \text{Vect}(X)$*

$$\mathcal{L}_{\bar{v}}^K D_{\mathbf{p}}^{\mathbf{I}} \Phi \in H_b^\infty(X \setminus Z, S \otimes \mathbf{I}; \mathbf{p});$$

in fact, for every $k, \ell \in \mathbf{N}_0$ with $\ell > d/2$, and $\varepsilon > 0$, there is a constant $c_k(\varepsilon) < \infty$ such that

$$\|\mathcal{L}_{\bar{v}}^K D_{\mathbf{p}}^{\mathbf{I}} \Phi\|_{H_b^k} \leq \varepsilon \|\bar{v}\|_{H^{k+1}} \|\Phi\|_{K^\ell} + c_k(\varepsilon) \|\bar{v}\|_{H^1} \|\Phi\|_{K^{k+\ell}}.$$

Proof. Evidently,

$$\begin{aligned} \|\mathcal{L}_{\bar{v}}^K D_{\mathbf{p}}^{\mathbf{I}} \Phi\|_{H_b^k} &\lesssim_k \|\nabla \bar{v} \otimes D_{\mathbf{p}}^{\mathbf{I}} \Phi\|_{H_b^k} + \|\bar{v} \otimes \nabla D_{\mathbf{p}}^{\mathbf{I}} \Phi\|_{H_b^k} \\ &= \|\nabla \bar{v} \otimes D_{\mathbf{p}}^{\mathbf{I}} \Phi\|_{H_b^k} + \|r^{-1} w_{\log}^{-1} \bar{v} \otimes r w_{\log} \nabla D_{\mathbf{p}}^{\mathbf{I}} \Phi\|_{H_b^k}. \end{aligned}$$

The summands on the right-hand side are estimated individually.

By [BW26, Proposition 4.26], $H_a^\ell \Gamma(X \setminus Z, S \otimes \mathbf{I}; 0, \mathbf{p}) \subseteq L^\infty \Gamma(X \setminus Z, S \otimes \mathbf{I}; \mathbf{p})$. Moreover, by definition of $\|\cdot\|_{K^k}$ and (3.53),

$$\|D_{\mathbf{p}}^{\mathbf{I}} \Phi\|_{H_a^{k+1}} \lesssim_k \|\Phi\|_{K^k}.$$

Therefore,

$$\|\nabla \bar{v} \otimes D_{\mathbf{p}}^{\mathbf{I}} \Phi\|_{H_b^k} \lesssim_k \sum_{m=0}^k \|\nabla \bar{v}\|_{H^{k-m}} \|D_{\mathbf{p}}^{\mathbf{I}} \Phi\|_{H_a^{m+\ell}} \lesssim_k \sum_{m=0}^k \|\bar{v}\|_{H^{k-m+1}} \|\Phi\|_{K^{m+\ell-1}}.$$

By (the proof of) the borderline L^2 Hardy inequality in dimension two,

$$\|r^{-1} w_{\log}^{-1} \bar{v}\|_{H_b^k} \lesssim_k \|\bar{v}\|_{H^{k+1}};$$

see, e.g., [DS13, Proposition 3.1]. Moreover, by [BW26, Proposition 4.28], $H_b^\ell \Gamma(X \setminus Z, S \otimes \mathbf{I}; \mathbf{p}) \subseteq r^{-1} L^\infty \Gamma(X \setminus Z, S \otimes \mathbf{I}; \mathbf{p})$. Finally, by definition of $\|\cdot\|_{K^k}$ and (3.53),

$$\|w_{\log} \nabla D_{\mathbf{p}}^{\mathbf{I}} \Phi\|_{H_b^k} \lesssim_k \|\Phi\|_{K^k}.$$

Therefore,

$$\|r^{-1} w_{\log}^{-1} \bar{v} \otimes r w_{\log} \nabla D_{\mathbf{p}}^{\mathbf{I}} \Phi\|_{H_b^k} \lesssim_k \sum_{m=0}^k \|\bar{v}\|_{H^{k-m+1}} \|w_{\log} \nabla D_{\mathbf{p}}^{\mathbf{I}} \Phi\|_{H_b^{m+\ell}} \lesssim_k \sum_{m=0}^k \|\bar{v}\|_{H^{k-m+1}} \|\Phi\|_{K^{m+\ell}}.$$

Combining the above estimates,

$$\|\mathcal{L}_{\bar{v}}^K D_{\mathbf{p}}^{\mathbf{I}} \Phi\|_{H_b^k} \lesssim_k \sum_{m=0}^k \|\bar{v}\|_{H^{k-m+1}} \|\Phi\|_{K^{m+\ell}}.$$

This implies the asserted estimate using interpolation as in [Ham79, §4.2; DL25, Lemma A.4]. $\blacksquare$

Proposition 3.55 (cf. [BW26, Remark 4.45]). Let $(\mathbf{p}, [\mathbf{I}]) \in \mathbf{Dir} \times \mathbf{Ram}$ and set $Z := \text{Br}(\mathbf{I})$. Choose a local infinitesimal uniformiser as in Proposition 3.34. For every $\Phi \in K^\infty(X \setminus Z, S \otimes \mathbf{I}; \mathbf{p})$ and $v \in T_{[\mathbf{I}]} \mathbf{Ram} = \Gamma(Z, NZ)$

$$\mathcal{L}_v^K \Phi \in H_a^\infty \Gamma(X \setminus Z, S \otimes \mathbf{I}; \mathbf{p});$$

moreover, for every $\Phi \in K^\infty(X \setminus Z, S \otimes \mathbf{I}; \mathbf{p})$ there is a unique $\mathbf{L}_\Phi \in \Gamma(Z, \text{Hom}_{\mathbb{A}}(\overline{NZ}, \check{S}_{\mathbf{p}}))$ such that for every $v \in T_{[\mathbf{I}]} \mathbf{Ram}$

$$\text{res}(\mathcal{L}_v^K \Phi) = \mathbf{L}_\Phi(v|_Z).$$

Proof. By Remark 3.40 and Proposition 3.52, $\nabla_v^{\text{Ram}} \mathbf{D}(\mathbf{p}, \mathbf{I}; \Phi) \in H_b^\infty \Gamma(X \setminus Z, S \otimes \mathbf{I}; \mathbf{p})$. Therefore, by Proposition 3.39 and Proposition 3.54, $\mathcal{L}_v^K \Phi \in H_a^\infty \Gamma(X \setminus Z, S \otimes \mathbf{I}; \mathbf{p})$.

A moment's thought reveals that $v \mapsto \text{res}(\mathcal{L}_v^K \Phi)$ is independent of the choice of infinitesimal uniformiser. Moreover, it is $C^\infty(Z)$ -linear and, therefore, defines $\mathbf{L}_\Phi \in \Gamma(Z, \text{Hom}(NZ, \check{S}_{\mathbf{p}}))$. Direct inspection of $D_{\mathbf{p}}^{\mathbf{I}}$, or rather the model operator $\hat{D}_{\mathbf{p}}^{\mathbf{I}}$, shows that $\mathbf{L}_\Phi$ is $\mathbb{A}$ -anti-linear. ■

By the method explained in Section 3.4, the above fibres assemble into the following tame Fréchet space bundle.

Definition 3.56. Consider the tame Fréchet space bundle

$$\mathbf{F}_{00} := w_{\log}^{-1} \mathbf{E}_0.$$

The universal bundle of admissible $\mathbb{Z}/2\mathbb{Z}$ spinors is the tame Fréchet subbundle

$$\mathbf{E}_{00} := \mathbf{E}_0 \cap \mathbf{D}^{-1} \mathbf{F}_{00} \subseteq \mathbf{E}_0;$$

indeed, the preceding discussion and direct inspection show that the atlas constructed in Definition 3.25 induces an atlas for $\mathbf{E}_{00}$; cf. Remark 3.27 (1). Of course, the tame Fréchet space structure on each fibre of $\mathbf{E}_{00}$ is the one defined in Proposition 3.51. Denote by

$$\mathbf{E}_{00}^* \subseteq \mathbf{E}_{00}$$

the open subset of **non-degenerate** admissible $\mathbb{Z}/2\mathbb{Z}$ spinors; that is: the subset consisting of those admissible $\mathbb{Z}/2\mathbb{Z}$ spinors $(\mathbf{p}, [\mathbf{I}]; \Phi) \in \mathbf{E}_{00}$ for which $\mathbf{L}_\Phi$ is injective. •

The crucial observation is that every non-degenerate admissible $\mathbb{Z}/2\mathbb{Z}$ harmonic spinor defines a residue condition.

Proposition 3.57. Assume that $\text{rk}_{\mathbb{A}} \check{S} = 2$. Denote by $f: \mathbf{E}_{00}^* \subseteq \mathbf{E}_{00} \rightarrow \mathbf{Dir} \times \mathbf{Ram}$ the projection map and by

$$\mathbf{V} \subseteq (p^* f)^* \check{S}$$

the subbundle such that for every $(\mathbf{p}, [\mathbf{I}]; \Phi) \in \mathbf{E}_{00}^*$

$$\mathbf{V}_{\mathbf{p}, [\mathbf{I}]; \Phi} = \text{im } \mathbf{L}_\Phi.$$

$(\mathbf{E}_{00}^*, f, \mathbf{V})$ is a family of local residue conditions that satisfies the hypothesis of Proposition 3.50.

Proof. Since $\mathbf{L}_\Phi$ is $\mathbb{A}$ -anti-linear and $\dim \text{im } \mathbf{L}_\Phi = \frac{1}{2} \text{rk } \check{S}$, this follows from [BW26, Example 4.44]. ■

3.8 Chiral adaptations

Throughout this subsection, assume that X is oriented and $\dim X = 0 \pmod{4}$. For every $\mathbf{p} \in \mathbf{Dir}$ and $[I] \in \mathbf{Ram}$ the volume form induces a chirality operator $\varepsilon := \gamma(\text{vol}_g)$. This decomposes $S = S^+ \oplus S^-$ and splits the entire discussion so far into positive and negative chiral parts. Most of this is entirely formal.

Definition 3.58. Here is the decomposition of the objects constructed in [Section 3.4](#):

- (1) Decompose the tame Fréchet space bundles E , F , and G into tame Fréchet subbundles

$$(3.59) \quad E = E^+ \oplus E^-, \quad F = F^+ \oplus F^-, \quad \text{and} \quad G = G^+ \oplus G^-$$

with fibres over $(\mathbf{p}, [I]) \in \mathbf{Dir} \times \mathbf{Ram}$ given by

$$\begin{aligned} E_{\mathbf{p}, I}^\pm &= H_a^\infty \Gamma(X \setminus \text{Br}(I), S^\pm \otimes I; \mathbf{p}), \\ F_{\mathbf{p}, I}^\pm &= H_b^\infty \Gamma(X \setminus \text{Br}(I), S^\pm \otimes I; \mathbf{p}), \quad \text{and} \\ G_{\mathbf{p}, I}^\pm &= \Gamma(\text{Br}(I), \check{S}_\mathbf{p}^\pm). \end{aligned}$$

- (2) Decompose the universal Dirac operator $D: E \rightarrow F$ into the **universal chiral operator** $D^\pm: E^\pm \rightarrow F^\mp$ such that

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}.$$

- (3) Decompose the universal residue map $\text{res}: E \rightarrow G$ into the **universal chiral residue map** $\text{res}^\pm: E^\pm \rightarrow G^\pm$. •

$$\begin{array}{ccc} E^\pm & \xrightarrow{D^\pm} & F^\mp \\ \text{res}^\pm \downarrow & & \\ G^\pm & & \end{array}$$

Figure 4: The universal chiral Dirac operator and residue map.

The partial covariant derivatives constructed in [Section 3.5](#) respect the above decomposition mostly, but not entirely.

Proposition 3.60.

- (1) *The restriction of the slice partial covariant derivatives ∇^{Sl} to $H_{\text{BG}}^{\text{Sl}}$ and V_∇ preserves the splittings (3.59).*
- (2) *In the situation of [Definition 3.38](#), the partial covariant derivatives ∇^{Ram} preserve the splittings (3.59); moreover, for every $(\mathbf{p}, [I]; \phi) \in E^\pm|_{\mathcal{U}}$ and $v \in T_{[I]}\mathbf{Ram}$*

$$\nabla_v^{\text{Ram}} D^\pm(\mathbf{p}, [I]; \phi) = [\mathcal{L}_v^K, D_\mathbf{p}^I] \phi.$$

Proof. (1) follows by direct computation. By direct inspection, κ_v defined in [Definition 3.14](#) commutes with ε . This implies (2). $\blacksquare$

Definition 3.61. (Local) chiral residue conditions can be imposed in families as follows:

(1) A family of chiral residue conditions $(B, f, \mathbf{R}^\pm)$ consists of:

- (a) a tame Fréchet manifold B ,
- (b) a tame smooth map $f: B \rightarrow \mathbf{Dir} \times \mathbf{Ram}$, and
- (c) a tame Fréchet space subbundle $\mathbf{R}^\pm \subseteq f^* \mathbf{G}^\pm$.

Set

$$\mathbf{E}_{\mathbf{R}^\pm}^\pm := (f^* \mathbf{res}^\pm)^{-1}(\mathbf{R}^\pm)$$

and denote the restriction of $f^* \mathbf{D}$ to $\mathbf{E}_{\mathbf{R}^\pm}^\pm$ by

$$\mathbf{D}_{\mathbf{R}^\pm}^\pm: \mathbf{E}_{\mathbf{R}^\pm}^\pm \rightarrow f^* \mathbf{F}^\mp.$$

(2) (a) Decompose the universal residue bundle $\check{\mathbf{S}}$ as

$$\check{\mathbf{S}} = \check{\mathbf{S}}^+ \oplus \check{\mathbf{S}}^-$$

into the **universal chiral residue bundles** $\check{\mathbf{S}}^\pm$ over $\mathbf{Br}$.

(b) A family of local chiral residue conditions $(\mathbf{B}, f, \mathbf{V}^\pm)$ consists of:

- (i) a tame Fréchet manifold $\mathbf{B}$,
- (ii) a tame smooth map $f: \mathbf{B} \rightarrow \mathbf{Dir} \times \mathbf{Ram}$, and
- (iii) a subbundle $\mathbf{V}^\pm \subseteq (p^* f)^* \check{\mathbf{S}}^\pm$.

Here $f^* p: f^* \mathbf{Br} \rightarrow \mathbf{B}$ is the pullback of the universal branching locus and $p^* f: f^* \mathbf{Br} \rightarrow \mathbf{Br}$ is the projection map.

(c) A family of local chiral residue conditions $(\mathbf{B}, f, \mathbf{V}^\pm)$ defines a family of chiral residue conditions $(\mathbf{B}, f, \mathbf{R}^\pm)$ such that for every $b \in \mathbf{B}$

$$\mathbf{R}_b^\pm := \Gamma(Z_b, \mathbf{V}^\pm|_{Z_b}) \subseteq \Gamma(Z_b, \check{\mathbf{S}}_{p_b}^\pm) \quad \text{with} \quad (p_b; Z_b, [I_b]) := f(b). \quad \bullet$$

It is immediate from the discussion in [Section 2.5](#) that the following variant of [Proposition 3.50](#) holds.

Proposition 3.62. *Let $(\mathbf{B}, f, \mathbf{V}^\pm)$ be a family of local chiral residue conditions such that for every $b \in \mathbf{B}$, $x \in Z_b$, and $v \in T_x Z_b \setminus \{0\}$*

$$\mathbf{V}_{(b,x)}^\pm \oplus J_{p_b} \gamma(v) \mathbf{V}_{(b,x)}^\pm = \check{\mathbf{S}}_{p_b,x}^\pm.$$

Denote by $(\mathbf{B}, f, \mathbf{R}^\pm)$ the corresponding family of chiral residue conditions. If $\alpha: f^ \mathbf{F}^\pm \rightarrow f^* \mathbf{F}^\mp$ is a map of tame Fréchet bundles of order zero, then*

$$\mathbf{D}_{\mathbf{R}^\pm}^\pm + \alpha: \mathbf{E}_{\mathbf{R}^\pm}^\pm \rightarrow f^* \mathbf{F}^\mp$$

is uniformly Fredholm. Moreover, for every $b_0 \in \mathbf{B}$ the maps $\pi: \mathbf{E}_{\mathbf{R}^\pm}^\pm|_{\mathcal{U}} \rightarrow \mathbf{Def}$ and $\iota: \mathbf{Ob} \hookrightarrow f^ \mathbf{F}^\mp|_{\mathcal{U}}$ in [Definition 3.49](#) can be chosen such that there is a compact subset $K \subseteq X$ such that for every $b \in \mathcal{U}$ the following hold:*

- (1) $K \subseteq X \setminus Z_b$;
- (2) if $\phi \in \mathbf{E}_{\mathbf{R}^\pm, b}^\pm$ has $\text{supp}(\phi) \cap K = \emptyset$, then $\pi(\phi) = 0$; and
- (3) $\text{supp}(\iota(o)) \subseteq K$ for every $o \in \text{Ob}_b$. ■

Finally, it remains to decompose the universal bundle of admissible $\mathbb{Z}/2\mathbb{Z}$ spinors and the resulting family of local residue conditions.

Definition 3.63. The tame Fréchet space bundles $\mathbf{E}_{00}$ and $\mathbf{F}_{00}$ defined in [Definition 3.56](#) decompose into

$$\mathbf{E}_{00} = \mathbf{E}_{00}^+ \oplus \mathbf{E}_{00}^- \quad \text{and} \quad \mathbf{F}_{00} = \mathbf{F}_{00}^+ \oplus \mathbf{F}_{00}^-$$

with $\mathbf{E}_{00}^\pm$ denoting the **universal bundle of admissible chiral $\mathbb{Z}/2\mathbb{Z}$ spinors**. The latter contains the open subset

$$\mathbf{E}_{00}^{\pm,*} \subseteq \mathbf{E}_{00}^\pm$$

of **non-degenerate** admissible chiral $\mathbb{Z}/2\mathbb{Z}$ spinors. For every $(\mathbf{p}, [\mathbf{I}]; \Phi) \in \mathbf{E}_{00}^{\pm,*}$, the map $\mathbf{L}_\Phi$ constructed in [Proposition 3.55](#) corestricts to

$$\mathbf{L}_\Phi^\pm \in \Gamma(Z, \text{Hom}_{\mathbb{A}}(\overline{NZ}, \check{S}_{\mathbf{p}}^\pm)).$$

The images define a family of local chiral residue conditions

$$(\mathbf{E}_{00}^{\pm,*}, f, V^\pm).$$

If $\text{rk}_{\mathbb{A}} \check{S}^\pm = 2$, this family satisfies the hypothesis of [Proposition 3.62](#). •

4 The constructions of the tame Fréchet manifold structures

Using the framework laid out in [Section 3](#), this section constructs tame Fréchet manifold structures on $\mathbf{PHarm}^*$, $\mathbf{PEig}^*$, and $\mathbf{PHarm}^{\pm,*}$, thereby establishing [Theorem 1.1](#), [Theorem 1.6](#), and [Theorem 1.7](#). The final two subsections derive [Theorem 1.10](#) and [Theorem 1.13](#).

4.1 Uniformly Fredholm maps

The results to be proved in this section refer to the following notion.

Definition 4.1. Let $\mathbf{X}, \mathbf{Y}$ be tame Fréchet manifolds. A tame smooth map $f: \mathbf{X} \rightarrow \mathbf{Y}$ is **uniformly Fredholm** if for every $x \in \mathbf{X}$ there are a tame Fréchet space $\mathbf{F}$, a chart $\phi: U \rightarrow \tilde{U} \subseteq \mathbf{F} \times \mathbb{R}^k$ of $\mathbf{X}$ with $x \in U$, a chart $\psi: V \rightarrow \tilde{V} \subseteq \mathbf{F} \times \mathbb{R}^\ell$ of $\mathbf{Y}$ with $f(x) \in V$, a smooth map $F: \tilde{U} \rightarrow \mathbb{R}^\ell$ such that

$$\psi \circ f \circ \phi^{-1}(y, z) = (y, F(y, z)). \quad \bullet$$

In this case, $k - \ell$ is independent of the choice of charts and defines the **index** of f as a locally constant map

$$\text{index } f: \mathbf{X} \rightarrow \mathbb{Z}.$$

Remark 4.2. A uniformly Fredholm map for which one can always choose $\ell = 0$ is nothing but a **uniform submersion** in the sense of [Glö16, p. 2].² Variations of the above notion are discussed in [Die, §2.2.3; DR22, §3.3]. ♣

The significance of uniformly Fredholm maps arises from the fact that they are compatible with transverse base change.

Proposition 4.3. *Let X , Y , and Z be tame Fréchet manifolds, $f: X \rightarrow Y$ uniformly Fredholm and $g: Z \rightarrow Y$ tame smooth. If f and g are **transverse**; that is: for every $(x, z) \in X \times Z$ with $f(x) = g(z) =: y$ the map $T_x f + T_z g: T_x X \oplus T_z Z \rightarrow T_y Y$ is surjective; then:*

- (1) *the fibre product*

$$g^*X := \{(x, z) \in X \times Z : f(x) = g(z)\}$$

is a tame Fréchet submanifold, and

- (2) *the map $g^*f: g^*X \rightarrow Z$ is uniformly Fredholm and $\text{index } g^*f = (\text{index } f) \circ f^*g$.*

Proof. It suffices to consider the situation in which F is a tame Fréchet space, X is an open subset of $F \times \mathbb{R}^k$, Y is an open subset of $F \times \mathbb{R}^\ell$, and $f(x_0, x_1) = (x_0, F(x_0, x_1))$. By direct inspection,

$$g^*X = \{(z, x_0, x_1) \in Z \times F \times \mathbb{R}^k : x_0 = \text{pr}_1 \circ g(z), F(\text{pr}_1 \circ g(z), x_1) = \text{pr}_2 \circ g(z)\}.$$

By the transversality assumption, at every $(z, x_0, x_1) \in g^*X$, the derivative of the map $(z, x_1) \mapsto F(\text{pr}_1 \circ g(z), x_1) - \text{pr}_2 \circ g(z)$ is surjective. Therefore, the assertion follows from the version of the regular value theorem stated in [Ham82, Part III Theorem 2.3.1]. ■

The hypothesis in Proposition 4.3 is generic according to the following parametric transversality result.

Proposition 4.4. *Let X , Y , and P be second-countable tame Fréchet manifolds, Z a second-countable (finite-dimensional) manifold, and $f: X \rightarrow Y$ uniformly Fredholm. If $G: P \times Z \rightarrow Y$ is transverse to f , in particular, if it is a **naive submersion**; that is: $T_{(p,z)}G: T_p P \oplus T_z Z \rightarrow T_{G(p,z)}Y$ is surjective for every $(p, z) \in P \times Z$, then the subset*

$$P^\natural := \{p \in P : g_p := G(p, -): Z \rightarrow Y \text{ is transverse to } f\}$$

is comeager.

The proof relies on the following observation.

Theorem 4.5 (Smale [Sma65]). *Let X and Y be second-countable tame Fréchet manifolds. If $f: X \rightarrow Y$ is uniformly Fredholm, then its set of regular values*

$$\text{RegVal}(f) := \{y \in Y : T_x f \text{ is surjective for every } x \in f^{-1}(y)\}$$

is comeager.

Proof. As pointed out by [DR22, Remark 3.4], the proof in [Sma65] carries over to the present situation. ■

²Glöckner does not use the adjective *uniform*.

Proof of Proposition 4.4. By Proposition 4.3, G^*X is a second-countable tame Fréchet submanifold, and $G^*f: G^*X \rightarrow P \times Z$ is uniformly Fredholm. Let $(p, z, x) \in G^*X$ and set $y := f(x)$. The Snake Lemma applied to

$$\begin{array}{ccccc} T_z Z \oplus T_x X & \hookrightarrow & T_p P \oplus T_z Z \oplus T_x X & \twoheadrightarrow & T_p P \\ T_z g_p - T_x f \downarrow & & \downarrow T_{(p,z)} G - T_x f & & \\ T_y Y & \xlongequal{\quad} & T_y Y & & \end{array}$$

yields an exact sequence

$$T_{(p,z,x)} G^*X \xrightarrow{T_{(p,z,x)}(\text{pr}_P \circ G^*f)} T_p P \twoheadrightarrow \text{coker}(T_z g_p - T_x f).$$

Therefore, $p \in P^\natural$ if and only if it is a regular value of $\text{pr}_P \circ G^*f: G^*X \rightarrow P$. Since Z is finite-dimensional, $\text{pr}_P \circ G^*f$ is uniformly Fredholm. Therefore, the assertion follows from Theorem 4.5. $\blacksquare$

Remark 4.6. There is a version of Theorem 4.5 for uniformly left semi-Fredholm maps [Qui70]. Using this the hypothesis that Z is finite-dimensional can be dropped. $\clubsuit$

4.2 The Nash–Moser submersion theorem

The proofs of Theorem 1.1, Theorem 1.6, and Theorem 1.7 use the following Nash–Moser version of the submersion theorem.

Theorem 4.7. *Let X and B be tame Fréchet manifolds, V a tame Fréchet space bundle over X , $\text{pr}_B: X \rightarrow B$ a uniform submersion, $s \in \Gamma(X, V)$ a tame smooth section, and ∇ a partial connection on V with respect to the vertical tangent bundle $TX/B := \ker T\text{pr}_B \subseteq TX$. If there is a finite rank vector bundle Def over X and a map of tame Fréchet space bundles $\pi: TX/B \twoheadrightarrow \text{Def}$ such that the thickening*

$$(4.8) \quad \begin{pmatrix} \nabla s \\ \pi \end{pmatrix}: TX/B \rightarrow V \oplus \text{Def}$$

is an isomorphism of tame Fréchet space bundles, then:

- (1) *the zero locus*

$$Z := s^{-1}(0) \subseteq X$$

is a tame Fréchet submanifold,

- (2) *the projection $\text{pr}_B: Z \rightarrow B$ is a uniform submersion, and*

- (3) *$TZ/B = \ker T(\text{pr}_B|_Z) \cong \text{Def}|_Z$.*

Proof. Since the statement is local, it suffices to consider the situation in which $X = U \times B$ is the product of open neighbourhoods of the origin in two tame Fréchet spaces, and V is a product bundle, whose fibre we continue to denote by V . In this case,

$$s: U \times B \rightarrow V$$

is a tame smooth map, and we may further assume that $(0, 0) \in s^{-1}(0)$.

The assumptions imply that $\text{Def} \cong \ker(\nabla s)$; let Def_0 denote the fibre over $(0, 0)$, and let $\pi_0: TU \rightarrow \text{Def}_0$ be a projection to this fibre. For an open neighbourhood of the origin $\mathbf{K} \subseteq \text{Def}_0$, consider the family of augmented maps

$$\bar{s}: \mathbf{X} \times \mathbf{K} \rightarrow \mathbf{V} \oplus \text{Def}_0$$

defined by $\bar{s}(x, k) := (s(x), \pi_0(x) - k)$, where $\pi_0(x)$ is understood by viewing $\mathbf{U} \subseteq TU$ as a subset. The covariant derivative

$$\nabla \bar{s} := (\nabla s, \pi_0): TX/\mathbf{B} \rightarrow \mathbf{V} \oplus \text{Def}_0$$

is an isomorphism; indeed, viewing $\text{Def} \subseteq TU$ as $\ker(\nabla s)$ as above, $\pi_0: \text{Def} \rightarrow \text{Def}_0$ is a map of finite-rank vector bundles that is the identity over the origin, hence we may assume it is an isomorphism after shrinking $\mathbf{U}$, $\mathbf{B}$, and $\mathbf{K}$.

Let $G: \mathbf{V} \rightarrow TU$ denote the restriction of the inverse of (4.8) to the first component in the codomain, which is a tame smooth family of right inverses of ∇s . Then

$$\bar{G}: \mathbf{V} \oplus \text{Def}_0 \rightarrow TX/\mathbf{B} \quad \text{with} \quad \bar{G}_{(u,b)}(v, q) := G_{(u,b)}v + \pi_0^{-1}(q - \pi_0 G_{(u,b)}v)$$

is a tame smooth family of inverses for $\nabla \bar{s}$. Moreover,

$$(T\bar{s})_{(u,b)}\bar{G}_{(u,b)}(v, q) = (v, q) - (\Gamma_{(u,b)}(\bar{G}_{(u,b)}(v, q))s, 0).$$

Thus $\bar{G}$ is an inverse of $T\bar{s}$ up to an $\bar{s}$ -quadratic error term. [Ham82, Part III Theorem 3.3.4] then applies to show that there is a tame smooth map $\Phi: \mathbf{B} \times \mathbf{K} \rightarrow \mathbf{X}$ whose value at (b, k) is the unique solution to

$$s(\Phi(b, k)) = 0 \quad \text{and} \quad \pi_0(\text{pr}_{\mathbf{U}}\Phi(b, k)) = k$$

in $\mathbf{X}$. By construction, $(\text{pr}_{\mathbf{B}}, \pi_0 \circ \text{pr}_{\mathbf{U}}) \circ \Phi = \text{id}$, hence Φ provides a chart on $\mathbf{Z} = s^{-1}(0)$ which is already of the form in Definition 4.1. This shows (1) and (2). The same composition in the second factor alone shows that $T\Phi(b, -): \text{Def}_0 \rightarrow TZ/\mathbf{B}$ is an isomorphism, which shows (3). ■

4.3 The universal space of non-degenerate $\mathbb{Z}/2\mathbb{Z}$ harmonic spinors

This subsection proves Theorem 1.1; that is: assuming that $\dim X = 3$ and $\text{rk } S = 4$, the space of non-degenerate $\mathbb{Z}/2\mathbb{Z}$ harmonic spinors

$$\text{Harm}^* := \{(\mathbf{p}, [\mathbf{I}]; \Phi) \in \mathbf{E}_0 : D_{\mathbf{p}}^{\mathbf{I}}\Phi = 0 \text{ and } \Phi \text{ is non-degenerate}\}$$

shall be equipped with the structure of a tame Fréchet manifold such that the projection map $\text{pr}_{\text{Dir}}: \text{Harm}^* \rightarrow \text{Dir}$ is uniformly Fredholm of index zero. Since

$$\text{PHarm}^* = \text{Harm}^*/\mathbb{R}^{\times},$$

this immediately implies Theorem 1.1. Of course, throughout the following four subsubsections it is assumed that $\dim X = 3$ and $\text{rk } S = 4$.

4.3.1 Local thickenings of the universal space of $\mathbb{Z}/2\mathbb{Z}$ harmonic spinors

The following is the essential step in the proof of [Theorem 1.1](#).

Proposition 4.9. *For every $(p_0, [I_0]; \Phi_0) \in \mathbf{Harm}^*$ there are an open subset $\mathcal{U} \subseteq E_{00}^*$ with $(p_0, [I_0]; \Phi_0) \in \mathcal{U}$ and a finite rank subbundle $\text{Ob} \subseteq f^*F_{00}|_{\mathcal{U}}$ such that:*

- (1) $\mathcal{V} := \text{pr}_{\text{Dir}}(\mathcal{U}) \subseteq \mathbf{Dir}$ is open,
- (2) the local thickening

$$\mathbf{Harm}^t := \{(p, [I]; \Phi) \in \mathcal{U} : D_p^I \Phi \in \text{Ob}\} \subseteq \mathcal{U}$$

is a tame Fréchet manifold, and

- (3) the projection map $\text{pr}_{\mathcal{V}} : \mathbf{Harm}^t \rightarrow \mathcal{V}$ is a uniform submersion.

Proof. The proof consists of five steps. Denote by (E_{00}^*, f, R) the residue condition from [Proposition 3.57](#). Let $(p_0, [I_0]; \Phi_0) \in \mathbf{Harm}^*$.

Step 1. *Choice of $\mathcal{U}$ and $\text{Ob} \subseteq F_{00}|_{\mathcal{U}}$.*

By [Proposition 3.50](#) and [Proposition 3.57](#), there are an open neighbourhood $(p_0, [I_0]; \Phi_0) \in \mathcal{U} \subseteq E_{00}^*$, finite rank vector bundles Def and Ob over $\mathcal{U}$, and maps of tame Fréchet space bundles $\pi : E_R|_{\mathcal{U}} \rightarrow \text{Def}$ and $\iota : \text{Ob} \hookrightarrow f^*F|_{\mathcal{U}}$ such that

$$\begin{pmatrix} D_R & \iota \\ \pi & 0 \end{pmatrix}$$

is invertible. Moreover, [Proposition 3.50](#) guarantees that ι can be chosen so that it factors through $j : \text{Ob} \hookrightarrow f^*F_{00}|_{\mathcal{U}}$; in fact, the $\iota(o)$ are always supported away from Z . Henceforth, Ob shall be regarded as a subbundle of $f^*F_{00}|_{\mathcal{U}}$.

Since $\text{pr}_{\text{Dir}} : E_{00}^* \rightarrow \mathbf{Dir}$ is a submersion, $\mathcal{V} := \text{pr}_{\text{Dir}}(\mathcal{U}) \subseteq \mathbf{Dir}$ is open; therefore, (1) holds. The task at hand is to prove that, after possibly shrinking $\mathcal{U}$, (2) and (3) hold with the help of [Theorem 4.7](#).

Step 2. *Setup for [Theorem 4.7](#).*

Consider the tame Fréchet space bundle

$$Q := f^*F_{00}/\text{Ob}$$

over $\mathcal{U}$. Define the section $s \in \Gamma(\mathcal{U}, Q)$ by

$$s(p, [I]; \Phi) := D_p^I \Phi \pmod{\text{Ob}_{p, [I]; \Phi}}.$$

Evidently, $\mathbf{Harm}^t = s^{-1}(0)$.

The vertical tangent bundle of $\mathcal{U}$ fits into the short exact sequence

$$f^*E_{00} \hookrightarrow T\mathcal{U}/\mathbf{Dir} \twoheadrightarrow f^*\mathcal{F}^{\text{Ram}}.$$

Choose a local infinitesimal uniformiser and define ∇^{Ram} as in [Definition 3.38](#). This splits the above short exact sequence and defines a partial covariant derivative on $\mathcal{Q}$ such that, for every $(\mathbf{p}, [I]; \Phi) \in \mathcal{U}$

$$(4.10) \quad \begin{aligned} (\nabla s)_{(\mathbf{p}, [I]; \Phi)} : \mathbf{E}_{00; \mathbf{p}, [I]} \oplus T_{[I]} \mathbf{Ram} &\rightarrow \mathcal{Q}_{\mathbf{p}, [I]; \Phi} \\ (\phi, v) &\mapsto D_{\mathbf{p}}^I \phi + (\nabla_v^{\text{Ram}} \mathbf{D})(\mathbf{p}, [I]; \Phi) \pmod{\text{Ob}_{\mathbf{p}, [I]; \Phi}}. \end{aligned}$$

It can be arranged that the subsets $\text{supp}(\nabla_v^{\text{Ram}} \mathbf{D})(\mathbf{p}, [I]; \Phi)$ are always disjoint from the compact subset in which the elements of Ob are supported.

The task at hand is to prove that, after possibly shrinking $\mathcal{U} \ni (\mathbf{p}_0, [I_0]; \Phi_0)$, the thickening

$$\begin{pmatrix} \nabla s \\ \pi \end{pmatrix}$$

is an isomorphism of tame Fréchet space bundles and thus the hypotheses of [Theorem 4.7](#) are satisfied. The latter directly implies [\(2\)](#) and [\(3\)](#).

Step 3. *Reduction to the invertibility of a lift.*

It is evident from [\(4.10\)](#) that ∇s lifts to

$$\widetilde{\nabla s} : f^*(\mathbf{E}_0 \oplus \mathcal{F}^{\text{Ram}}) \rightarrow \tilde{\mathcal{Q}} := f^* \mathbf{F}|_{\mathcal{U}/\text{Ob}}$$

such that the following diagram commutes

$$\begin{array}{ccc} f^*(\mathbf{E}_{00} \oplus \mathcal{F}^{\text{Ram}}) & \xrightarrow{(\nabla s, \pi)} & \mathcal{Q} \oplus \text{Def} \\ \downarrow & & \downarrow \\ f^*(\mathbf{E}_0 \oplus \mathcal{F}^{\text{Ram}}) & \xrightarrow{(\widetilde{\nabla s}, \pi)} & \tilde{\mathcal{Q}} \oplus \text{Def}. \end{array}$$

By [Proposition 3.52](#) and [Remark 3.40](#), for every $(\mathbf{p}, [I]; \Phi) \in \mathbf{E}_{00}$, $(\nabla_v \mathbf{D})(\mathbf{p}, [I]; \Phi) \in \mathbf{F}_{00; \mathbf{p}, [I]}$. Therefore, if $(\phi, v) \in \mathbf{E}_{0; \mathbf{p}, [I]} \oplus \mathcal{F}_{\mathbf{p}, [I]}^{\text{Ram}}$ and $\psi \in \mathbf{F}_{00; \mathbf{p}, [I]}$ satisfy

$$D_{\mathbf{p}}^I \phi + \nabla_v^{\text{Ram}} \mathbf{D}(\mathbf{p}, [I]; \Phi) = \psi \pmod{\text{Ob}_{\mathbf{p}, [I]; \Phi}},$$

then $D_{\mathbf{p}}^I \phi \in \mathbf{F}_{00; \mathbf{p}, [I]}$. As a consequence, if $(\widetilde{\nabla s}, \pi)$ is an isomorphism of tame Fréchet space bundles, then $(\nabla s, \pi)$ is an isomorphism of tame Fréchet space bundles.

The next two steps verify that $(\widetilde{\nabla s}, \pi)$ is an isomorphism of tame Fréchet space bundles, after possibly shrinking $\mathcal{U}$.

Step 4. *A slight change of perspective.*

Define $\Theta_{\text{ext}}, \Theta_K : f^*(\mathbf{E}_0 \oplus \mathcal{F}^{\text{Ram}}) \rightarrow \mathbf{E}_R$ by

$$\begin{aligned} \Theta_{\text{ext}}(\mathbf{p}, [I], \Phi; \phi, v) &:= (\mathbf{p}, [I]; \Phi; \phi - \text{ext}(\mathbf{L}_{\Phi} v)) \quad \text{and} \\ \Theta_K(\mathbf{p}, [I], \Phi; \phi, v) &:= (\mathbf{p}, [I]; \Phi; \phi - \mathcal{L}_{\Phi}^K \Phi). \end{aligned}$$

By the discussion in [Section 3.6](#), Θ_{ext} is an isomorphism of tame Fréchet space bundles with inverse

$$\Theta_{\text{ext}}^{-1}(\mathbf{p}, [\mathbf{I}], \Phi; \psi) := (\mathbf{p}, [\mathbf{I}], \Phi; \psi - \text{ext}(\text{res}(\psi)), -\mathbf{L}_{\Phi}^{-1} \text{res}(\psi)).$$

By direct computation,

$$\Theta_{\text{ext}}^{-1} \circ \Theta_K(\mathbf{p}, [\mathbf{I}], \Phi; \phi, v) = (\mathbf{p}, [\mathbf{I}], \Phi; \phi - \Psi_{\Phi}(v), v)$$

with

$$\Psi_{\Phi}(v) := \mathcal{L}_{\bar{v}}^K \Phi - \text{ext}(\mathbf{L}_{\Phi} v).$$

By construction, $\text{res}(\Psi_{\Phi}(v)) = \text{res}(\mathcal{L}_{\bar{v}}^K \Phi) - \text{res}(\text{ext}(\mathbf{L}_{\Phi} v)) = 0$. Evidently, $\Theta_{\text{ext}}^{-1} \circ \Theta_K$ is an isomorphism of tame Fréchet space bundles; hence, so is Θ_K .

From [Proposition 3.39](#) and [\(4.10\)](#) it follows that

$$\widetilde{\mathbf{V}}_S \circ \Theta_K^{-1} = \mathbf{D}_R + \mathbf{P} \pmod{\text{Ob}}$$

with the perturbation $\mathbf{P}$ defined by

$$\mathbf{P}(\mathbf{p}, [\mathbf{I}], \Phi; \psi) := (\mathbf{p}, [\mathbf{I}], \Phi; \mathcal{L}_{\bar{v}_{\Phi}(\psi)}^K D_{\mathbf{p}}^{\mathbf{I}} \Phi) \quad \text{and} \quad \bar{v}_{\Phi}(\psi) := \bar{v}(\mathbf{p}, [\mathbf{I}], -\mathbf{L}_{\Phi}^{-1} \text{res}(\psi)).$$

Step 5. Proof of invertibility.

By construction, the thickening

$$\overline{\mathbf{D}}_R := \begin{pmatrix} \mathbf{D}_R \\ \pi \end{pmatrix} \pmod{\text{Ob}}$$

is an isomorphism of tame Fréchet space bundles and its inverse $\overline{\mathbf{D}}_R^{-1}$ satisfies tame estimates of the form

$$\|\overline{\mathbf{D}}_R^{-1}(\psi, \delta)\|_{H_a^{k+1}} \lesssim \|\psi\|_{H_b^k} + \|\delta\|.$$

By [Proposition 3.54](#), $\mathbf{P}$ is a small perturbation in the sense of [\[DL25, Appendix A\]](#). Therefore, as a consequence of [\[DL25, Proposition A.5\]](#), $\overline{\mathbf{D}}_R + \mathbf{P}$ is invertible. $\blacksquare$

4.3.2 Transversality of the obstruction map

Proposition 4.11. *In the situation of [Proposition 4.9](#) the **obstruction map** $\text{ob}: \text{Harm}^t \rightarrow \text{Ob}$ defined by*

$$\text{ob}(\mathbf{p}, [\mathbf{I}]; \Phi) := D_{\mathbf{p}}^{\mathbf{I}} \Phi$$

is transverse to the zero section.

Proof of [Proposition 4.11](#). Let $(\mathbf{p}, [\mathbf{I}]; \Phi) \in \text{ob}^{-1}(0) \subseteq \text{Harm}^*$. By [Proposition 3.30](#), for every $\dot{\mathbf{V}} \in \Omega^1(X, \mathfrak{o}_{\text{Cl}}(S))$, $\dot{\mathbf{p}}_{\mathbf{V}} = (0; 0; \dot{\mathbf{V}}; 0) \in T_{(\mathbf{p}, [\mathbf{I}])} \text{Sl}$ and

$$\nabla_{\dot{\mathbf{p}}_{\mathbf{V}}}^{\text{Sl}} \mathbf{D}(\mathbf{p}, [\mathbf{I}]; \Phi) = \sum_{i=1}^3 \gamma(e_i) \dot{\mathbf{V}}_{e_i} \Phi.$$

There is a compact subset $K \subseteq X \setminus Z$ with non-empty interior on which Φ does not vanish. By [Remark 3.7](#) and [Lemma 4.12](#), stated after this proof, for every $\psi \in \Gamma(X \setminus Z, S \otimes \mathbb{I})$ with $\text{supp}(\psi) \subseteq K$ there is a $\dot{\nabla}$ such that

$$\sum_{i=1}^3 \gamma(e_i) \dot{\nabla}_{e_i} \Phi = \psi.$$

Since $\text{pr}_{\text{Dir}}: \mathbf{Harm}^t \rightarrow \mathbf{Dir}$ is a submersion, there is a lift $\Omega^1(X, \mathfrak{o}_{C\ell}(S)) \rightarrow T_{(\mathbf{p}, [\mathbb{I}]; \Phi)} \mathbf{Harm}^t$. From the above it is clear that the composition

$$\Omega^1(X, \mathfrak{o}_{C\ell}(S)) \hookrightarrow T_{(\mathbf{p}, [\mathbb{I}]; \Phi)} \mathbf{Harm}^t \xrightarrow{T_{(\mathbf{p}, [\mathbb{I}]; \Phi)} \text{ob}} T_{(\mathbf{p}, [\mathbb{I}]; \Phi; 0)} \text{Ob} \twoheadrightarrow \text{Ob}_{(\mathbf{p}, [\mathbb{I}]; \Phi)}$$

is surjective. Therefore, ob is transverse to the zero section. ■

Lemma 4.12. *For every $\phi \in \mathbb{H} \setminus \{0\}$ and $\psi \in \mathbb{H}$ there are $\xi, \eta, \zeta \in \text{Im } \mathbb{H}$ such that*

$$i\phi\xi + j\phi\eta + k\phi\zeta = \psi.$$

Proof. By direct inspection, it is possible to choose $\tilde{\xi}, \tilde{\eta}, \tilde{\zeta} \in \text{Im } \mathbb{H}$ such that $i\tilde{\xi} + j\tilde{\eta} + k\tilde{\zeta} = \psi\phi^{-1}$. Set $\xi := \phi^{-1}\tilde{\xi}\phi$, $\eta := \phi^{-1}\tilde{\eta}\phi$, $\zeta := \phi^{-1}\tilde{\zeta}\phi$. ■

Proof of Theorem 1.1. It is an immediate consequence of [Proposition 4.9](#), [Proposition 4.11](#), and [[Ham82](#), Part III, Theorem 2.3.1] that $\mathbf{Harm}^* \subseteq \mathbf{E}_{00}^*$ is a tame Fréchet submanifold and that $\text{pr}_{\text{Dir}}: \mathbf{Harm}^* \rightarrow \mathbf{Dir}$ is uniformly Fredholm.

By [[BW26](#), Corollary 2.30], the family of residue conditions introduced in [Proposition 3.57](#) is Lagrangian. As a consequence $\text{rk Def} = \text{rk Ob}$ and, therefore, pr_{Dir} has index zero. This directly implies [Theorem 1.1](#). ■

Remark 4.13. The proof of [Theorem 1.1](#) identifies the vertical tangent spaces of $\mathbf{Harm}^*$: indeed, if $(\mathbf{p}, [\mathbb{I}]; \Phi) \in \mathbf{Harm}^*$, then identifying $\mathbf{E}_{0, \mathbf{p}, [\mathbb{I}]} \oplus T_{[\mathbb{I}]} \mathbf{Ram} = \mathbf{E}_{\mathbf{R}, \mathbf{p}, [\mathbb{I}]; \Phi}$

$$\ker T_{(\mathbf{p}, [\mathbb{I}]; \Phi)} \text{pr}_{\text{Dir}} = \ker D_{R_\Phi}. \quad \clubsuit$$

Of course, the corresponding vertical tangent space of $\mathbf{PHarm}^*$ is $\ker D_{R_\Phi} / \mathbb{R} \langle \Phi \rangle$.

4.3.3 A transversality criterion

Here is how to verify whether or not a smooth map $\Gamma: B \rightarrow \mathbf{Dir}$ is transverse to $\text{pr}_{\text{Dir}}: \mathbf{PHarm}^* \rightarrow \mathbf{Dir}$.

Definition 4.14. Let $(\mathbf{p}, [\mathbb{I}]) \in \mathbf{Dir} \times \mathbf{Ram}$ and $\phi, \psi \in \Gamma(X \setminus \text{Br}(\mathbb{I}), S \otimes \mathbb{I})$.

- (1) The **stress-energy tensor** $T_{\phi, \psi} \in \Gamma(X \setminus Z, \text{Hom}(S^2 TX, \mathbb{R}))$ is defined by

$$T_{\phi, \psi}(v, w) := -\frac{1}{8}(\langle \gamma(v) \nabla_w \phi, \psi \rangle + \langle \gamma(w) \nabla_v \phi, \psi \rangle + \langle \gamma(v) \nabla_w \psi, \phi \rangle + \langle \gamma(w) \nabla_v \psi, \phi \rangle).$$

(2) Define $\mu(\phi, \psi) \in \Omega^1(X \setminus Z, \mathfrak{o}_{\text{C}\ell}(S))$ by

$$\mu(\phi, \psi) := \frac{1}{2} \bar{\gamma}^*(\phi \langle \psi, - \rangle + \psi \langle \phi, - \rangle)$$

with $\bar{\gamma}^*$ denoting the adjoint of the linear map $\bar{\gamma}: T^*X \otimes \mathfrak{o}_{\text{C}\ell}(S) \rightarrow \text{Sym}(S)$ defined by $\bar{\gamma}(\alpha \otimes \xi) := \gamma(\alpha^\sharp) \xi$. •

Proposition 4.15. *Let B be a manifold and $\Gamma: B \rightarrow \mathbf{Dir}$ a smooth map. The following are equivalent:*

- (1) Γ is transverse to $\text{pr}_{\mathbf{Dir}}: \mathbf{PHarm}^* \rightarrow \mathbf{Dir}$.
- (2) For every $b \in B$ and $(\mathbf{p}, [\mathbf{I}]; \Phi) \in \mathbf{Harm}^*$ with $\Gamma(b) = \mathbf{p}$ the map

$$\delta_\Phi: T_b B \rightarrow (\ker D_{R_\Phi})^*$$

defined by

$$\delta_\Phi(v) \kappa := \langle T_{\Phi, \kappa}, T_{\mathbf{p}} \text{pr}_{\mathbf{Met}} \circ T_b \Gamma(v) \rangle_{L^2} + \langle \mu(\Phi, \kappa), \text{pr}_{V_\nabla} \circ T_b \Gamma(v) \rangle_{L^2}$$

is surjective. Here R_Φ denotes the residue condition induced by Φ .

Proof. Let $b \in B$ and $(\mathbf{p}, [\mathbf{I}]; \Phi) \in \mathbf{Harm}^*$ with $\Gamma(b) = \mathbf{p}$. In the construction of $\mathbf{Harm}^t$, Ob can be chosen such that the L^2 orthogonal projection Π onto $\ker D_{R_\Phi}$ induces an isomorphism $\text{Ob}_{(\mathbf{p}, [\mathbf{I}]; \Phi)} \cong \ker D_{R_\Phi}$. In particular, as a consequence of the construction of $\mathbf{Harm}^*$, Γ and $\text{pr}_{\mathbf{Dir}}$ are transverse at b and $(\mathbf{p}, [\mathbf{I}]; \Phi)$ precisely if the map

$$\begin{aligned} T_b B &\rightarrow \ker D_{R_\Phi} \\ v &\mapsto \Pi \nabla_{T_b \Gamma(v)} \mathbf{D}(\mathbf{p}, [\mathbf{I}]; \Phi) \end{aligned}$$

is surjective. By direct inspection this is equivalent to δ_Φ being surjective; cf. [Mai97, Proof of Proposition 4.1] or [AD25, Proof of Lemma 3.2]. Evidently, the $V_{\mathcal{E}}$ component of $T_b \Gamma(v)$ does not contribute. ■

Remark 4.16. The (dual of the) condition (2) is a **Petri condition**; cf. [Wen23, §5; DW23, Definition 1.1.11]. Here are some comments on the usefulness of the contributions to δ_Φ to verify this condition:

- (1) The map $\mu(\Phi, -): \Gamma(X \setminus Z, S \otimes \mathbf{I}) \rightarrow \Omega^1(X \setminus Z, \mathfrak{o}_{\text{C}\ell}(S))$ factors through a linear map $S \otimes \mathbf{I} \rightarrow \text{Hom}(TX, \mathfrak{o}_{\text{C}\ell}(S))|_{X \setminus Z}$ which is injective on the dense open subset $X \setminus \Phi^{-1}(0)$ by Lemma 4.12. Of course, this is what enabled the proof of Proposition 4.11 and makes the second contribution to δ_Φ quite useful.
- (2) The map $T_{\Phi, -}: \Gamma(X \setminus Z, S \otimes \mathbf{I}) \rightarrow \Gamma(X \setminus Z, \text{Hom}(S^2 TX, \mathbb{R}))$ factors through $J^1(S \otimes \mathbf{I}) \rightarrow \text{Hom}(S^2 TX, \mathbb{R})|_{X \setminus Z}$. If $T_{\Phi, \Phi} = 0$, then g is conformally flat by [Mai97, Proof of Theorem 1.2]. Therefore, the first contribution to δ_Φ might not always be useful. However, it is quite possible that for generic metrics it is. Maybe, the methods developed by [GK26] can be brought to bear on this.

The above issue also is what stands in the way of establishing a version of Theorem 1.1 for $\mathbb{Z}/2\mathbb{Z}$ harmonic spinors on Dirac bundles arising from a (topological) spin structure on X . ♣

4.3.4 The universal zero locus

Here is an observation to the effect that generic non-degenerate $\mathbb{Z}/2\mathbb{Z}$ harmonic spinors do not vanish outside of the branching locus. This observation essentially already appears in [HP24, Proof of Corollary 1.12] and is repeated here for the readers' convenience.

Proposition 4.17. *The **universal zero locus** of non-degenerate $\mathbb{Z}/2\mathbb{Z}$ harmonic spinors*

$$\mathbf{Z}_{\mathbf{PHarm}^*} := \{(\mathbf{p}, [\mathbf{I}]; [\Phi], x) \in \mathbf{PHarm}^* \times X : x \notin \text{Br}(\mathbf{I}), \Phi(x) = 0\}$$

is a tame Fréchet submanifold of $\mathbf{PHarm}^ \times X$ and $\text{pr}_{\mathbf{PHarm}^*} : \mathbf{Z}_{\mathbf{PHarm}^*} \rightarrow \mathbf{PHarm}^*$ has index -1 .*

Proof. There is a rank 4 vector bundle $\mathcal{S}$ over $\mathring{\mathbf{X}}_{\mathbf{PHarm}^*} := \{(\mathbf{p}, [\mathbf{I}]; [\Phi], x) \in \mathbf{PHarm}^* \times X : x \notin \text{Br}(\mathbf{I})\}$ whose fibre over $(\mathbf{p}, [\mathbf{I}]; [\Phi], x)$ is $\text{Hom}(\mathbb{R}\langle\Phi\rangle, S_x \otimes \mathbf{I}_x)$. Evaluation defines a section $\text{ev} \in \Gamma(\mathring{\mathbf{X}}_{\mathbf{PHarm}^*}, \mathcal{S})$. If ev is transverse to the zero section, then, by [Ham82, Part III, Theorem 2.3.1], $\mathbf{Z}_{\mathbf{PHarm}^*}$ is a tame Fréchet submanifold and $\text{index } \text{pr}_{\mathbf{PHarm}^*} = \dim X - \text{rk } \mathcal{S} = -1$.

To prove that ev is transverse to the zero section, it suffices to prove that for every $(\mathbf{p}, [\mathbf{I}]; [\Phi], x) \in \mathbf{Z}_{\mathbf{PHarm}^*}$ and $\psi \in (S \otimes \mathbf{I})_x$ there are $\phi \in H_a^\infty \Gamma(X \setminus Z, S \otimes \mathbf{I}; R_\Phi; \mathbf{p})$ and $\dot{\mathbf{p}} \in T_{\mathbf{p}} \mathbf{Dir}$ such that

$$\phi(x) = \psi \quad \text{and} \quad D_{\mathbf{p}}^{\mathbf{I}} \phi + \nabla_{\dot{\mathbf{p}}} \mathbf{D}(\mathbf{p}, [\mathbf{I}]; \Phi) = 0.$$

To prove this, it suffices to prove that

$$\begin{aligned} \ker \text{ev}_x \oplus T_{\mathbf{p}} \mathbf{Dir} &\rightarrow H_b^\infty(X \setminus Z, S \otimes \mathbf{I}; \mathbf{p}) \\ (\phi, \dot{\mathbf{p}}) &\mapsto D_{\mathbf{p}}^{\mathbf{I}} \phi + \nabla_{\dot{\mathbf{p}}} \mathbf{D}(\mathbf{p}, [\mathbf{I}]; \Phi) \end{aligned}$$

is surjective. Here $\text{ev}_x : H_a^\infty \Gamma(X \setminus Z, S \otimes \mathbf{I}; R_\Phi; \mathbf{p}) \rightarrow S_x \otimes \mathbf{I}_x$ denotes the evaluation map. The restriction of $D_{\mathbf{p}}^{\mathbf{I}}$ to $\ker \text{ev}_x$ is Fredholm and $\text{coker } D_{\mathbf{p}}^{\mathbf{I}}|_{\ker \text{ev}_x} \cong (\ker D_{R_\Phi} \oplus D_{\mathbf{p}}^{\mathbf{I}} \widetilde{S_x \otimes \mathbf{I}_x})^*$ for every choice of lift $\widetilde{S_x \otimes \mathbf{I}_x}$ of $S_x \otimes \mathbf{I}_x$ along ev_x . With this in mind, the surjectivity of the above map follows by the argument employed in the proof of Proposition 4.11. $\blacksquare$

The above might be somewhat interesting in view of [DW21; Yan25a; Yan25b].

4.4 The universal space of non-degenerate $\mathbb{Z}/2\mathbb{Z}$ eigenspinors

Assuming that $\dim X = 3$ and $\text{rk } S = 4$, consider the universal space of non-degenerate $\mathbb{Z}/2\mathbb{Z}$ eigenspinors

$$\mathbf{Eig}^* := \{(\mathbf{p}, [\mathbf{I}]; \Phi, \lambda) \in \mathbf{E}_{00}^* \times \mathbb{R} : D_{\mathbf{p}}^{\mathbf{I}} \Phi = \lambda \Phi\}.$$

To prove that $\mathbf{Eig}^*$ is a submanifold and $\text{pr}_{\mathbf{Dir}} : \mathbf{Eig}^* \rightarrow \mathbf{Dir}$ is uniformly Fredholm of index 1, one proceeds as in Section 4.3.

Proposition 4.18. *For every $(\mathbf{p}_0, [\mathbf{I}_0]; \Phi_0, \lambda_0) \in \mathbf{Eig}^*$ there are an open subset $\mathcal{U} \subseteq \mathbf{E}_{00}^* \times \mathbb{R}$ with $(\mathbf{p}_0, [\mathbf{I}_0]; \Phi_0, \lambda_0) \in \mathcal{U}$ and a finite rank subbundle $\text{Ob} \subseteq \text{pr}_1^* f^* \mathbf{F}_{00}|_{\mathcal{U}}$ such that:*

- (1) $\mathcal{V} := \text{pr}_{\mathbf{Dir}}(\mathcal{U}) \subseteq \mathbf{Dir}$ is open,

(2) the local thickening

$$\mathbf{Eig}^t := \{(\mathbf{p}, [\mathbf{I}]; \Phi, \lambda) \in \mathcal{U} : (D_{\mathbf{p}}^{\mathbf{I}} - \lambda)\Phi \in \text{Ob}\} \subseteq \mathcal{U}$$

is a tame Fréchet manifold, and

(3) the projection map $\text{pr}_{\mathcal{V}} : \mathbf{Eig}^t \rightarrow \mathcal{V}$ is a uniform submersion.

Proof. The proof of [Proposition 4.9](#) carries over with very minor modifications. In particular, Ob is constructed using [Proposition 3.50](#) applied to $\mathbf{D} - \lambda$ and s is defined by

$$s(\mathbf{p}, [\mathbf{I}]; \Phi, \lambda) := (D_{\mathbf{p}}^{\mathbf{I}} - \lambda)\Phi \pmod{\text{Ob}_{\mathbf{p}, [\mathbf{I}]; \Phi, \lambda}}. \quad \blacksquare$$

Proposition 4.19. *In the situation of [Proposition 4.18](#) the obstruction map $\text{ob} : \mathbf{Eig}^t \rightarrow \text{Ob}$ defined by*

$$\text{ob}(\mathbf{p}, [\mathbf{I}]; \Phi, \lambda) := (D_{\mathbf{p}}^{\mathbf{I}} - \lambda)\Phi$$

is transverse to the zero section. ■

Proof of [Theorem 1.6](#). As a consequence of [Proposition 4.18](#), [Proposition 4.19](#), and [[Ham82](#), Part III, Theorem 2.3.1], it follows that $\mathbf{Eig}^* \subseteq \mathbf{E}_{00}^* \times \mathbb{R}$ is a tame Fréchet manifold and that $\text{pr}_{\text{Dir}} : \mathbf{Eig}^* \rightarrow \text{Dir}$ is uniformly Fredholm of index 1. This implies the assertion. ■

Here is the analogue of the transversality criterion [Proposition 4.15](#).

Proposition 4.20. *Let B be a manifold and $\Gamma : B \rightarrow \text{Dir}$ a smooth map. The following are equivalent:*

- (1) Γ is transverse to $\text{pr}_{\text{Dir}} : \mathbf{PEig}^* \rightarrow \text{Dir}$.
- (2) For every $b \in B$ and $(\mathbf{p}, [\mathbf{I}]; [\Phi], \lambda) \in \mathbf{PEig}^*$ with $\Gamma(b) = \mathbf{p}$ the map

$$\delta_{\Phi, \lambda} : T_b B \times \mathbb{R} \rightarrow (\ker(D_{R_{\Phi}} - \lambda))^*$$

defined by

$$\delta_{\Phi, \lambda}(v, t)\kappa := \langle T_{\Phi, \kappa}, T_{\mathbf{p}} \text{pr}_{\text{Met}} \circ T_b \Gamma(v) \rangle_{L^2} + \langle \mu(\Phi, \kappa), \text{pr}_{V_{\nabla}} \circ T_b \Gamma(v) \rangle_{L^2} - t \langle \Phi, \kappa \rangle_{L^2}$$

is surjective. Here R_{Φ} denotes the residue condition induced by Φ . ■

Remark 4.21. It should be observed that $\text{tr}_g T_{\Phi, \kappa} = -\frac{1}{2}\lambda \langle \Phi, \kappa \rangle$ and, therefore, $T_{\Phi, -}$ cannot vanish if $\lambda \neq 0$. ♣

Finally, as in [Proposition 4.17](#), generic non-degenerate $\mathbb{Z}/2\mathbb{Z}$ eigenspinors do not vanish outside of the branching locus.

Proposition 4.22. *The universal zero locus of non-degenerate $\mathbb{Z}/2\mathbb{Z}$ eigenspinors*

$$\mathbf{Z}_{\mathbf{PEig}^*} := \{(\mathbf{p}, [\mathbf{I}]; [\Phi], \lambda, x) \in \mathbf{PEig}^* \times X : x \notin \text{Br}(\mathbf{I}), \Phi(x) = 0\}$$

is a tame Fréchet submanifold of $\mathbf{PEig}^ \times X$ and $\text{pr}_{\mathbf{PEig}^*} : \mathbf{Z}_{\mathbf{PEig}^*} \rightarrow \mathbf{PEig}^*$ has index -1 .* ■

Finally, it should be pointed out that it is straightforward to verify that zero is a regular value of the projection map $\lambda : \mathbf{PEig}^* \rightarrow \mathbb{R}$.

4.5 The universal space of non-degenerate chiral $\mathbb{Z}/2\mathbb{Z}$ harmonic spinors

Proof of Theorem 1.7. The proof of Theorem 1.1 carries over with cosmetic modifications and using Proposition 3.62 instead of Proposition 3.50. ■

Here is the analogue of the transversality criterion Proposition 4.15.

Proposition 4.23. *Let B be a manifold and $\Gamma: B \rightarrow \mathbf{Dir}$ a smooth map. The following are equivalent:*

- (1) Γ is transverse to $\mathrm{pr}_{\mathbf{Dir}}: \mathbf{PHarm}^{\pm,*} \rightarrow \mathbf{Dir}$.
- (2) For every $b \in B$ and $(\mathbf{p}, [\mathbf{I}]; \Phi) \in \mathbf{Harm}^{\pm,*}$ with $\Gamma(b) = \mathbf{p}$ the map

$$\delta_\Phi: T_b B \rightarrow (\ker D_{R_\Phi^\mp}^\mp)^*$$

defined by

$$\delta_\Phi(v)\kappa := \langle T_{\Phi,\kappa}, T_{\mathbf{p}}\mathrm{pr}_{\mathbf{Met}} \circ T_b\Gamma(v) \rangle_{L^2} + \langle \mu(\Phi, \kappa), \mathrm{pr}_{V_\nabla} \circ T_b\Gamma(v) \rangle_{L^2}$$

is surjective. Here $R_\Phi^\pm$ denotes the chiral residue condition induced by Φ . ■

Remark 4.24. The reader should beware that the codomain of δ_Φ is the dual of the kernel of the adjoint of $D_{R_\Phi^\pm}^\pm$. In the situation considered in Proposition 4.15, the relevant operator is self-adjoint and this distinction does not arise. ♣

4.6 The universal space of non-degenerate $\mathbb{Z}/2\mathbb{Z}$ harmonic 1-forms

This subsection proves Theorem 1.10. Throughout this subsection, $\dim X = 3$ and $S = \mathbb{R} \oplus T^*X$. The map $\iota_{\mathrm{HdR}}: \mathbf{Met} \rightarrow \mathbf{Dir}$, alluded to before the statement of Theorem 1.10, assigns to every Riemannian metric $g \in \mathbf{Met}$ the Dirac bundle structure on S with respect to which the Clifford multiplication is given by

$$\gamma(v)(f, \alpha) := (-i_v \alpha, v^\flat f + *(v^\flat \wedge \alpha))$$

and the spin connection is induced by the Levi-Civita connection. The pullback

$$\mathbf{Harm}_{\Omega^1}^* := \iota_{\mathrm{HdR}}^* \mathbf{Harm}^*$$

is the universal space of non-degenerate $\mathbb{Z}/2\mathbb{Z}$ harmonic 1-forms on X . Indeed, if $(g, [\mathbf{I}]; f, \alpha) \in \mathbf{Harm}_{\Omega^1}^*$, then

$$(4.25) \quad d^* df = d^*(df + *d\alpha) = 0$$

and, therefore, $df = 0$ by integration by parts and using that (f, α) has vanishing residue. Since $\mathrm{Br}(\mathbf{I}) \neq \emptyset$, an assumption that is tacitly made, $f = 0$. Therefore $([\mathbf{I}], \alpha)$ is a $\mathbb{Z}/2\mathbb{Z}$ harmonic 1-form with respect to g . Conversely and evidently, if $([\mathbf{I}], \alpha)$ is a non-degenerate $\mathbb{Z}/2\mathbb{Z}$ harmonic 1-form with respect to g , then $(g, [\mathbf{I}]; 0, \alpha) \in \mathbf{Harm}_{\Omega^1}^*$.

Here is some preparation regarding the finite rank vector bundle $\mathcal{H}^1 \rightarrow \mathbf{Met} \times \mathbf{Ram}$ mentioned in Theorem 1.10.

Proposition 4.26. *For every $[I] \in \mathbf{Ram}$ the double cover $\dot{\pi}: \tilde{X}^\circ := \{v \in I : |v| = 1\} \rightarrow X \setminus Z$ extends to a smooth branched double cover $\pi: \tilde{X} \rightarrow X$.*

Proof. The Euclidean line bundle I determines a double cover $\widetilde{SNZ}$ of the unit sphere bundle SNZ . The latter is the boundary of the unit disc bundle DNZ . The question is whether $\widetilde{SNZ}$ also is the boundary of a disc bundle $\widetilde{DNZ}$ that is a branched double cover of DNZ . The obstruction to the existence of $\widetilde{DNZ}$ is $w_2(NZ) \in H^2(Z, \{\pm 1\})$. The latter, however, is the restriction $PD([Z])|_Z$ and vanishes by [Remark 3.10](#). ■

Definition 4.27 (cf. [\[BW26, Example 3.48\]](#)). For every $(g, [I]) \in \mathbf{Met} \times \mathbf{Ram}$ the residue bundle $\check{S}$ over $Z := \text{Br}(I)$ contains the $\mathbb{A}$ -linear subbundle

$$\check{S}_D := N^*Z \otimes_{\mathbb{A}} NZ^{-1/2}.$$

These form a family of local residue conditions $(\mathbf{Met} \times \mathbf{Ram}, \iota_{\text{HdR}} \times \text{id}_{\mathbf{Ram}}, \check{S}_D)$. Denote the associated family of residue conditions by $(\mathbf{Met} \times \mathbf{Ram}, \iota_{\text{HdR}} \times \text{id}_{\mathbf{Ram}}, R_D)$. •

This is an elliptic Lagrangian family of local residue conditions. As a consequence of Teleman's Lipschitz Hodge theory [\[Tel83\]](#), the associated kernels form a vector bundle.

Proposition 4.28. *For every $(g, [I]) \in \mathbf{Met} \times \mathbf{Ram}$ the following hold:*

- (1) *The kernel of $D_{R_D, g}^I$ consists precisely of the L^2 harmonic I -twisted 1-forms; that is:*

$$\ker D_{R_D, g}^I = \mathcal{H}^1(g, [I]) := \{\alpha \in L^2 \Omega^1(X \setminus Z, I) : d\alpha = d^* \alpha = 0\}$$

- (2) *Pulling back along $\pi: \tilde{X} \rightarrow X$ induces an isomorphism*

$$\mathcal{H}^1(g, [I]) \cong H_{\text{dR}}^1(\tilde{X})^-.$$

Here $H_{\text{dR}}^1(\tilde{X})^-$ denotes the subspace of $H_{\text{dR}}^1(\tilde{X})$ which is anti-invariant under the sheet-swapping involution τ .

Proof. If $(f, \alpha) \in \ker D_{R_D, g}^I$, then $f = 0$ by the argument following [\(4.25\)](#). This proves that $\ker D_{R_D, g}^I \subseteq \mathcal{H}^1(g, [I])$. Conversely, if $\alpha \in \mathcal{H}^1(g, [I])$, then by [\[Don21a, \(4.1\)\]](#) it has an expansion of the form

$$\alpha = \text{Re}(Az^{-1/2}dz) + \text{higher order terms}.$$

Here z is a local complex normal coordinate and, globally, $A \in \Gamma(Z, NZ^{-1/2})$. Therefore, $(0, \alpha) \in H_a^\infty(X \setminus \text{Br}(I), S \otimes I)$ and $\text{res}(0, \alpha) \in R_D$. This proves [\(1\)](#).

The proof of [\(2\)](#) is a variation of the proof of [\[Wan93, Proposition 5\]](#). The metric π^*g degenerates along the ramification locus $\pi^{-1}(\text{Br}(I))$. This is a consequence of the fact that near the ramification locus π is locally modelled on $(x, z) \mapsto (x, z^2)$. There is a Lipschitz map $\sigma: \tilde{X} \rightarrow X$ which is: (1) modelled on $(x, z) \mapsto (x, z^2/|z|)$ near the ramification locus, (2) invariant under τ , (3) homotopic to π , and (4) such that $\tilde{g} := \sigma^*g$ is a Lipschitz Riemannian metric in the sense of [\[Tel83, §3\]](#). A moment's thought shows that $\mathcal{H}^1(g, [I])$ pulls back under σ precisely to the space of τ -anti-invariant L^2 harmonic forms with respect to $\tilde{g}$ in the sense of [\[Tel83, \(4.4\)\]](#). Therefore, the assertion follows from the version of the Hodge theorem stated in [\[Tel83, Theorem 4.1\]](#). ■

Corollary 4.29. *The vector spaces $\mathcal{H}^1(g, [I])$ form a finite rank subbundle of $(\iota_{\text{HdR}} \times \text{id}_{\text{Ram}})^* E \rightarrow \text{Met} \times \text{Ram}$.* ■

It turns out that the local residue conditions defined in [Proposition 3.57](#) and [Definition 4.27](#) are related as follows.

Proposition 4.30. *If $(g, [I]; \alpha) \in \text{Harm}_{\Omega^1}^*$, then*

$$\text{im } L_\alpha = \check{S}_D.$$

Proof. As explained in [\[Don21a, \(4.1\)\]](#), α has an expansion of the form

$$\alpha = \text{Re}(Bz^{1/2}dz) + \text{higher order terms}.$$

Here z is a local complex normal coordinate and, globally, $B \in \Gamma(Z, NZ^* \otimes_{\mathbb{A}} NZ^{-1/2})$. A moment's thought shows that up to a constant factor B is nothing but L_α . As a consequence, $\text{im } L_\alpha \subseteq \check{S}_D$. Since $\text{rk } L_\alpha = \text{rk } \check{S}_D$, this inclusion is an identity. ■

Proof of Theorem 1.10. Since ι_{HdR} is not transverse to $\text{pr}_{\text{Dir}}: \text{Harm}^* \rightarrow \text{Dir}$, a priori $\text{Harm}_{\Omega^1}^*$ might not be a tame Fréchet submanifold of $\iota_{\text{HdR}}^* E_{00}$.

Let $(g_0, [I_0]; \alpha) \in \text{Harm}_{\Omega^1}^*$. Consider a local thickening Harm^t containing $(\iota_{\text{HdR}}(g_0), [I_0]; \alpha)$ as in [Proposition 4.9](#). Since $\text{pr}_{\mathcal{V}}: \text{Harm}^t \rightarrow \mathcal{V}$ is a uniform submersion, $\iota_{\text{HdR}}^* \text{Harm}^t \subseteq \iota_{\text{HdR}}^* E_{00}^*$ is a tame Fréchet submanifold and the projection map $\iota_{\text{HdR}}^* \text{Harm}^t \rightarrow \iota_{\text{HdR}}^{-1}(\mathcal{V})$ is a uniform submersion. Evidently, $\iota_{\text{HdR}}^* \text{Harm}^t \supseteq \text{Harm}_{\Omega^1}^* \cap \iota_{\text{HdR}}^* \mathcal{U}$.

By the preceding propositions the thickening Harm^t can be chosen such that for every $(g, [I]; \alpha) \in \iota_{\text{HdR}}^* \text{Harm}^t$ the L^2 orthogonal projection $\text{Ob}_{g, [I]; \alpha}: \mathcal{H}^1(g, [I]) \rightarrow \mathcal{H}^1(g, [I])$ is injective. By construction, if $(g, [I]; \alpha) \in \iota_{\text{HdR}}^* \text{Harm}^t$, then

$$d^* \alpha = 0 \quad \text{and} \quad * d\alpha \in \text{Ob}_{g, [I]; \alpha}.$$

However, the L^2 orthogonal projection of $*d\alpha$ onto $\mathcal{H}^1(g, [I])$ vanishes by integration by parts, since α has vanishing residue. As a consequence the obstruction map $\iota_{\text{HdR}}^* \text{ob}$ vanishes. Therefore, $\iota_{\text{HdR}}^* \text{Harm}^t$ is an open neighborhood of $(g_0, [I_0]; \alpha)$. This proves (1).

In light of [Remark 4.13](#), (2) holds as a consequence of [Proposition 4.28](#) and [Proposition 4.30](#). ■

4.7 The universal space of non-degenerate $\mathbb{Z}/2\mathbb{Z}$ harmonic self-dual 2-forms

This subsection proves [Theorem 1.13](#). Throughout this subsection, $\dim X = 4$, X is oriented, and $S = T^*X \oplus \mathbb{R} \oplus \Lambda^+ T^*X$. The map $\iota_{\text{HdR}}: \text{Met} \rightarrow \text{Dir}$, alluded to before the statement of [Theorem 1.13](#), assigns to every Riemannian metric $g \in \text{Met}$ the Dirac bundle structure on S with respect to which the Clifford multiplication is given by

$$\gamma(v)(\alpha; f, \beta) := (v^\flat f - \sqrt{2}i_v \beta; -i_v \alpha, \sqrt{2}(v^\flat \wedge \alpha)^+)$$

and the spin connection is induced by the Levi-Civita connection. The pullbacks

$$\text{PHarm}_{\Omega^1}^* := \iota_{\text{HdR}}^* \text{PHarm}^{+,*} \quad \text{and} \quad \text{PHarm}_{\Omega^+}^* := \iota_{\text{HdR}}^* \text{PHarm}^{-,*}$$

are the universal moduli spaces of non-degenerate $\mathbb{Z}/2\mathbb{Z}$ harmonic 1-forms and self-dual 2-forms respectively.

Definition 4.31 (cf. [BW26, Example 3.48]). For every $(g, [\mathbf{I}]) \in \mathbf{Met} \times \mathbf{Ram}$ the chiral residue bundle $\check{S}^\pm$ over $Z := \text{Br}(\mathbf{I})$ contains the $\mathbb{A}$ -linear subbundle

$$\check{S}_D^+ := N^*Z \otimes_{\mathbb{A}} NZ^{-1/2} \quad \text{or} \quad \check{S}_D^- := (T^*Z \wedge N^*Z)^+ \otimes_{\mathbb{A}} NZ^{-1/2}$$

respectively. These form a family of local chiral residue conditions $(\mathbf{Met} \times \mathbf{Ram}, \iota_{\text{HdR}} \times \text{id}_{\mathbf{Ram}}, \check{S}_D^\pm)$. Denote the associated family of chiral residue conditions by $(\mathbf{Met} \times \mathbf{Ram}, \iota_{\text{HdR}} \times \text{id}_{\mathbf{Ram}}, \mathbf{R}_D^\pm)$. •

$\check{S}_D^+ \oplus \check{S}_D^-$ is an elliptic Lagrangian family of local residue conditions and the associated kernels form vector bundles.

Proposition 4.32. For every $(g, [\mathbf{I}]) \in \mathbf{Met} \times \mathbf{Ram}$ the following hold:

- (1) The kernel of $D_{R_D^\pm, g}^{\mathbf{I}}$ consists precisely of the L^2 harmonic $\mathbf{I}$ -twisted 1-forms and self-dual 2-forms respectively; that is:

$$\begin{aligned} \ker D_{R_D^+, g}^{\mathbf{I}} &= \mathcal{H}^1(g, \mathbf{I}) := \{\alpha \in L^2 \Omega^1(X \setminus Z, \mathbf{I}) : d\alpha = d^* \alpha = 0\} \quad \text{and} \\ \ker D_{R_D^-, g}^{\mathbf{I}} &= \mathcal{H}^+(g, \mathbf{I}) := \{\beta \in L^2 \Omega^+(X \setminus Z, \mathbf{I}) : d\beta = d^* \beta = 0\}. \end{aligned}$$

- (2) Pulling back along $\pi: \tilde{X} \rightarrow X$ induces isomorphisms

$$\mathcal{H}^1(g, \mathbf{I}) \cong H_{\text{dR}}^1(\tilde{X})^- \quad \text{and} \quad \mathcal{H}^+(g, \mathbf{I}) \cong H_{\text{dR}}^+(\tilde{X})^-.$$

Here $H_{\text{dR}}^\bullet(\tilde{X})^-$ denotes the subspace of $H_{\text{dR}}^\bullet(\tilde{X})$ which is anti-invariant under the sheet-swapping involution τ and $H_{\text{dR}}^+(\tilde{X})^- \subseteq H_{\text{dR}}^2(\tilde{X})^-$ denotes a maximal subspace on which the intersection form is positive-definite.

Proof. The proof is analogous to that of Proposition 4.28 using the fact that a $\beta \in \mathcal{H}^+(g, \mathbf{I})$ has an expansion of the form

$$\beta = \text{Re}(Az^{-1/2}dw \wedge dz) + \text{higher order terms}.$$

Here w is a local complex coordinate on Z and z is a local complex normal coordinate. ■

Corollary 4.33. The vector spaces $\mathcal{H}^1(g, \mathbf{I})$ and $\mathcal{H}^+(g, \mathbf{I})$ form finite rank subbundles of $(\iota_{\text{HdR}} \times \text{id}_{\mathbf{Ram}})^* \mathbf{E}^\pm \rightarrow \mathbf{Met} \times \mathbf{Ram}$ respectively. ■

Proposition 4.34. If $(g, [\mathbf{I}]; \alpha) \in \mathbf{Harm}_{\Omega^1}^*$ (or $(g, [\mathbf{I}]; \beta) \in \mathbf{Harm}_{\Omega^+}^*$), then

$$\text{im } L_\alpha = \check{S}_D^+ \quad (\text{or} \quad \text{im } L_\beta = \check{S}_D^-).$$

Proof. The proof is analogous to that of Proposition 4.30 using that

$$\beta = \text{Re}(Bz^{1/2}dw \wedge dz) + \text{higher order terms}.$$

Here w is a local complex coordinate on Z and z is a local complex normal coordinate and, globally, $B \in \Gamma(Z, (T^*Z \wedge NZ^*)^+ \otimes_{\mathbb{A}} NZ^{-1/2})$. ■

Proof of Theorem 1.13. The proof is analogous to that of Theorem 1.10 using the above ingredients and cosmetic modifications to track the chirality. ■

Remark 4.35. If $(g, [I]; \alpha) \in \text{Harm}_{\Omega^1}^*$, then the index of $D_{R_{\alpha}^+, g}^I$ can be determined as follows. If X is a closed oriented 4-manifold, then

$$b^1(X) - b^0(X) - b^+(X) = -\frac{1}{2}(\chi(X) + \sigma(X)).$$

By [GS99, Lemma 7.1.7],

$$\chi(\tilde{X}) = 2\chi(X) - \chi(Z) \quad \text{and} \quad \sigma(\tilde{X}) = 2\sigma(X) - \frac{1}{2}[Z] \cdot [Z].$$

If Z is not orientable, then $[Z] \cdot [Z]$ needs to be interpreted accordingly. The orientation local systems $\mathfrak{o}$ of TZ and NZ agree. Therefore, there is a well-defined Euler class $\mathfrak{e}(NZ) \in H^2(Z, \mathfrak{o})$ and $[Z] \in H_2(Z, \mathfrak{o})$ and both can be paired.

As a consequence of Proposition 4.32, Proposition 4.34, and the above,

$$\begin{aligned} \text{index } D_{R_{\alpha}^+, g}^{I,+} &= \dim \mathcal{H}^1(g, I) - \dim \mathcal{H}^+(g, I) \\ &= -\frac{1}{2}(\chi(\tilde{X}) + \sigma(\tilde{X})) + \frac{1}{2}(\chi(X) + \sigma(X)) \\ &= -\frac{1}{2}(\chi(X) + \sigma(X)) + \frac{1}{2}\chi(Z) + \frac{1}{4}[Z] \cdot [Z]. \end{aligned}$$

In fact, since α is non-degenerate $[Z] \cdot [Z] = 0$. Therefore, in the case under consideration, the correction term in Remark 1.9 is $\delta(S, I, \Phi) = \frac{1}{2}\chi(Z)$.

Finally, here is how to obtain the index formula in Remark 1.9 for $\text{Br}(I) = \emptyset$. As in [AHS78, Proof of Theorem 6.1], for the purpose of determining the index formula, there is no loss in assuming that X is spin. Denote by $\$$ the *real* spinor bundle. There is some quaternionic line bundle L such that $S = \$ \otimes_{\mathbb{H}} L$. By the Atiyah–Singer index formula [AS68, Theorem 5.3; LM89, Part III Theorem 13.10],

$$\text{index } D^+ = -\langle \hat{A}(X) \text{ch}_2(L), [X] \rangle = \frac{1}{4}\sigma(X) + \langle c_2(L), [X] \rangle.$$

The sign is a consequence of the real and complex chirality operators differing by a factor of -1 . It remains to determine $\langle c_2(L), [X] \rangle$. Since $p_1(S^\pm) = -c_2(\$^\pm \otimes_{\mathbb{C}} L) = -2(c_2(\$^\pm) + c_2(L))$ and $c_2(\$^\pm) = -\frac{1}{4}p_1(TX) \pm \frac{1}{2}e(TX)$,

$$\begin{aligned} c_2(L) &= -\frac{1}{2}p_1(S^\pm) - c_2(\$^\pm) \\ &= -\frac{1}{2}p_1(S^\pm) + \frac{1}{4}p_1(TX) \mp \frac{1}{2}e(TX) \\ &= -\frac{1}{2}p_1(S^\pm) + \frac{3}{4}\sigma(X) \mp \frac{1}{2}e(TX). \end{aligned}$$

The above combines to the stated contribution. ♣

Finally, here is a straightforward construction of non-degenerate $\mathbb{Z}/2\mathbb{Z}$ harmonic self-dual 2-forms on Kähler surfaces.

Example 4.36. Let X be a Kähler surface and $q \in H^0(X, K_X^{\otimes 2})$ transverse to zero. The latter defines a holomorphic branched double cover $\tilde{X} := \{(x, v) \in K_X : v^2 = q(x)\} \rightarrow X$. This induces a ramified Euclidean line bundle $\mathbf{I}$ with $Z := \text{Br}(\mathbf{I}) = q^{-1}(0)$ and $\text{Re } \sqrt{q} \in \Omega^+(X \setminus Z, \mathbf{I})$ is a $\mathbb{Z}/2\mathbb{Z}$ harmonic self-dual 2-form.

It is not difficult to find X and q as above. By Bertini's theorem, if $|2K_X|$ is base-point free, then a general $q \in H^0(X, K_X^{\otimes 2})$ is transverse to zero. A concrete example is a smooth hypersurface $X_d \subseteq \mathbb{C}P^3$ of degree $d \geq 4$ and a general section of $K_{X_d}^{\otimes 2} = \mathcal{O}_{X_d}(2d - 8)$. $\spadesuit$

By [Theorem 1.13](#), the $\mathbb{Z}/2\mathbb{Z}$ harmonic self-dual 2-forms obtained above persist under small deformations of the metric on X .

A Smoothing operators

The following appendix supplies the missing ingredient in the proof of [Proposition 3.51](#); that is: it constructs a family of smoothing operators for the scale of norms $(\|\cdot\|_{\dot{K}^k} : k \in \mathbb{N}_0)$ defined by

$$\|\phi\|_{\dot{K}^k} := \|\phi\|_{H_b^{k+2}} + \|\dot{\nabla}\phi\|_{H_b^{k+1}} + \|\log(r)\dot{D}\phi\|_{H_b^{k+1}} + \|\dot{\nabla}\log(r)\dot{D}\phi\|_{H_b^k}.$$

Proposition A.1. *There is a family of smoothing operators $(S_\varepsilon : \varepsilon \in (0, 1))$ on $L^2\Gamma(NZ \setminus Z, \dot{S} \otimes \dot{\mathbf{I}})$ such that for every $\varepsilon \in (0, 1)$, $\phi \in L^2\Gamma(NZ \setminus Z, \dot{S} \otimes \dot{\mathbf{I}})$, and $k, \ell \in \mathbb{N}_0$ with $k \geq \ell$,*

$$(A.2) \quad \|S_\varepsilon \phi\|_{\dot{K}^k} \lesssim_{k, \ell} \varepsilon^{-k+\ell} \|\phi\|_{\dot{K}^\ell},$$

$$(A.3) \quad \|(S_\varepsilon - 1)\phi\|_{\dot{K}^\ell} \lesssim_{k, \ell} \varepsilon^{k-\ell} \|\phi\|_{\dot{K}^k},$$

$$(A.4) \quad \varepsilon \|\partial_\varepsilon S_\varepsilon \phi\|_{\dot{K}^k} \lesssim_{k, \ell} \varepsilon^{-k+\ell} \|\phi\|_{\dot{K}^\ell}, \quad \text{and}$$

$$(A.5) \quad \varepsilon \|\partial_\varepsilon S_\varepsilon \phi\|_{\dot{K}^\ell} \lesssim_{k, \ell} \varepsilon^{k-\ell} \|\phi\|_{\dot{K}^k}.$$

As explained in [\[BW26, §3.4\]](#), Fubini's theorem and spectral theory imply that the Hilbert space $L^2\Gamma(NZ \setminus Z, \dot{S} \otimes \dot{\mathbf{I}})$ decomposes as

$$L^2\Gamma(NZ \setminus Z, \dot{S} \otimes \dot{\mathbf{I}}) = L^2((0, \infty), r dr, L^2(F, \underline{S} \otimes \underline{\mathbf{I}})) = \bigoplus_{(\lambda, \mu) \in \sigma} L^2((0, \infty), r dr; E_{\lambda, \mu}).$$

Here $\sigma \subseteq \mathbb{R}^2$ is a discrete subset and the $E_{\lambda, \mu}$ are finite-dimensional as in [\[BW26, Proposition 3.24\]](#). Moreover, by [\[BW26, Proposition 4.8\]](#), with respect to this decomposition

$$\|\phi\|_{H_b^k}^2 \asymp_k \sum_{(\lambda, \mu) \in \sigma} \sum_{\ell=0}^k \langle (\lambda, \mu) \rangle^{2\ell} \|(r \partial_r)^{k-\ell} \phi_{\lambda, \mu}\|_{r^{-1/2} L^2}^2$$

and

$$\|\dot{\nabla}\phi\|_{H_b^k}^2 \asymp_k \sum_{(\lambda, \mu) \in \sigma} \left(\langle \mu \rangle^2 \|\phi_{\lambda, \mu}\|_{H_b^k}^2 + \left\| \frac{\lambda}{r} \phi_{\lambda, \mu} \right\|_{H_b^k}^2 + \|\partial_r \phi_{\lambda, \mu}\|_{H_b^k}^2 \right).$$

The model Dirac operator does not quite preserve $E_{\lambda,\mu}$ but rather the direct sum $E_{\lambda,\mu} \oplus E_{-(\lambda+1),-\mu}$, on which it is given by

$$\mathring{D}^{\lambda,\mu} := \begin{pmatrix} \mu & J(\partial_r + \frac{\lambda+1}{r}) \\ J(\partial_r - \frac{\lambda}{r}) & -\mu \end{pmatrix}.$$

(The case $(\lambda, \mu) = (-1/2, 0)$ requires a bit more care as explained in [BW26, Definition 3.25], but this is inconsequential for the remaining discussion.)

The proof of Proposition 3.52, applied to the model operator, gives

$$\|\log(r) \mathring{\nabla} \phi\|_{H_b^{k+1}} \lesssim_k \|\phi\|_{\dot{K}^k}.$$

Together with [BW26, Proposition 4.8] and the above formula for $\mathring{D}^{\lambda,\mu}$, this yields

$$\begin{aligned} \|\phi\|_{\dot{K}^k}^2 \asymp_k \sum_{(\lambda,\mu) \in \sigma} & \left(\|\phi_{\lambda,\mu}\|_{H_b^{k+2}}^2 + \|\partial_r \phi_{\lambda,\mu}\|_{H_b^{k+1}}^2 + \|\frac{\lambda}{r} \phi_{\lambda,\mu}\|_{H_b^{k+1}}^2 + \|\log(r)(\partial_r - \frac{\lambda}{r})\phi_{\lambda,\mu}\|_{H_b^{k+1}}^2 \right. \\ & \left. + \|\partial_r \log(r)(\partial_r - \frac{\lambda}{r})\phi_{\lambda,\mu}\|_{H_b^k}^2 + \|\frac{\lambda}{r} \log(r)(\partial_r - \frac{\lambda}{r})\phi_{\lambda,\mu}\|_{H_b^k}^2 \right). \end{aligned}$$

Defining (S_ε) as the composition of a spectral cutoff at ε^{-1} in λ and μ , as in [Par23, Appendix B; BW26, §4.5], and the following smoothing operators in the radial direction establishes Proposition A.1.

Proposition A.6. *On $L^2((0, \infty), r dr; \mathbb{C})$ define*

$$\|\phi\|_{H_b^k} := \sum_{\ell=0}^k \|(r \partial_r)^\ell \phi\|_{r^{-1/2} L^2}$$

and for every $k \in \mathbb{N}_0$ and $\lambda \in \mathbb{R}$ set

$$\begin{aligned} \|\phi\|_{\dot{K}_\lambda^k} &:= \|\phi\|_{H_b^{k+2}} + \|\partial_r \phi\|_{H_b^{k+1}} + \|\frac{\lambda}{r} \phi\|_{H_b^{k+1}} + \|\log(r)(\partial_r - \frac{\lambda}{r})\phi\|_{H_b^{k+1}} \\ &\quad + \|\partial_r \log(r)(\partial_r - \frac{\lambda}{r})\phi\|_{H_b^k} + \|\frac{\lambda}{r} \log(r)(\partial_r - \frac{\lambda}{r})\phi\|_{H_b^k}. \end{aligned}$$

There is a family of smoothing operators $(S_\varepsilon : \varepsilon \in (0, 1))$ satisfying the analogues of (A.2), (A.3), (A.4), and (A.5).

The family (S_ε) in Proposition A.6 is constructed by **Mellin convolution** with a family of smoothing kernels σ_ε ; that is:

$$(S_\varepsilon \phi)(r) := (\sigma_\varepsilon *_{\mathcal{M}} \phi)(r) := \int_0^\infty \sigma_\varepsilon(r/s) \phi(s) \frac{ds}{s}.$$

The Mellin convolution $*_{\mathcal{M}}$ on $(0, \infty)$ is related to the (usual) convolution $*$ on $\mathbb{R}$ via the isometry $T : L^2((0, \infty), r dr; \mathbb{C}) \rightarrow L^2(\mathbb{R}; \mathbb{C})$ defined by

$$T(f)(x) := e^x f(e^x);$$

indeed

$$(f *_{\mathcal{M}} g)(r) = T^{-1}(Tf * Tg)(r).$$

Denote by $\mathcal{F} : L^2(\mathbb{R}; \mathbb{C}) \rightarrow L^2(\mathbb{R}; \mathbb{C})$ the Fourier transform. The **Mellin transform**

$$\mathcal{M} : L^2((0, \infty), r dr; \mathbb{C}) \rightarrow L^2(\mathbb{R}; \mathbb{C})$$

is defined by

$$\mathcal{M} := \mathcal{F} \circ T.$$

In particular,

$$\mathcal{M}(f *_{\mathcal{M}} g) = \mathcal{M}(f) \mathcal{M}(g).$$

Combining the identities $T(r\partial_r)T^{-1} = \partial_x - 1$ and $\mathcal{F}\partial_x\mathcal{F}^{-1} = 2\pi i\xi$ shows that for every $\phi \in C^\infty((0, \infty), \mathbb{C})$ the norms are expressed in terms of the Mellin transform by

$$(A.7) \quad \|\phi\|_{H_b^k} = \sum_{\ell=0}^k \|(r\partial_r)^\ell \phi\|_{r^{-1/2}L^2} \asymp_k \|\langle \xi \rangle^k \mathcal{M}(\phi)\|_{L^2}.$$

The following is the key to [Proposition A.6](#).

Lemma A.8. *Let $\chi \in C_c^\infty(\mathbb{R}, [0, 1])$ and $\rho \in C_c^\infty(\mathbb{R}, [0, 1])$ such that*

$$\chi|_{[-1, 1]} = 1, \quad \text{supp}(\chi) \subseteq [-2, 2], \quad \rho(0) = 1, \quad \text{and} \quad \text{supp}(\rho) \subseteq [-1, 1].$$

Define the family of smoothing kernels $(\sigma_\varepsilon : \varepsilon \in (0, 1))$ by

$$\sigma_\varepsilon(r) := \mathcal{M}^{-1}(\chi_\varepsilon)(r)\rho(\log(r))$$

with $\chi_\varepsilon(\xi) := \chi(\varepsilon\xi)$. The following hold:

(1) *For every $\varepsilon \in (0, 1)$, $k \in \mathbb{N}_0$, and $\xi \in \mathbb{R}$*

$$(A.9) \quad |\mathcal{M}(\sigma_\varepsilon)(\xi)| \lesssim_{\rho, k} \varepsilon^{-k} \langle \xi \rangle^{-k},$$

$$(A.10) \quad |\mathcal{M}(\sigma_\varepsilon)(\xi) - 1| \lesssim_{\rho, k} \varepsilon^k \langle \xi \rangle^k,$$

$$(A.11) \quad |\varepsilon \partial_\varepsilon \mathcal{M}(\sigma_\varepsilon)(\xi)| \lesssim_{\chi, \rho, k} \varepsilon^{-k} \langle \xi \rangle^{-k}, \quad \text{and}$$

$$(A.12) \quad |\varepsilon \partial_\varepsilon \mathcal{M}(\sigma_\varepsilon)(\xi)| \lesssim_{\chi, \rho, k} \varepsilon^k \langle \xi \rangle^k.$$

(2) *For every $\varepsilon \in (0, 1]$, $k \in \mathbb{Z}$, and $\xi \in \mathbb{R}$*

$$|\varepsilon^{-1} \mathcal{M}(\log(r)\sigma_\varepsilon)(\xi)| \lesssim_{\chi, \rho, k} \varepsilon^k \langle \xi \rangle^k \quad \text{and} \quad |\partial_\varepsilon \mathcal{M}(\log(r)\sigma_\varepsilon)(\xi)| \lesssim_{\chi, \rho, k} \varepsilon^k \langle \xi \rangle^k.$$

Proof. (1) By direct computation, using $\int_{\mathbb{R}} \hat{\rho} = \rho(0) = 1$,

$$\mathcal{M}(\sigma_\varepsilon) = \chi_\varepsilon * \hat{\rho}, \quad \mathcal{M}(\sigma_\varepsilon) - 1 = (\chi_\varepsilon - 1) * \hat{\rho}, \quad \text{and} \quad \partial_\varepsilon \mathcal{M}(\sigma_\varepsilon) = \partial_\varepsilon \chi_\varepsilon * \hat{\rho}.$$

Since $\rho \in C_c^\infty(\mathbb{R}) \subseteq \mathcal{S}(\mathbb{R})$, $\hat{\rho} \in \mathcal{S}(\mathbb{R})$ and, therefore,

$$|\hat{\rho}(\xi)| \lesssim_{\rho, k} \langle \xi \rangle^{-k}.$$

By Young's inequality and since $\|\chi_\varepsilon\|_{L^\infty} \leq 1$, $\|\varepsilon \partial_\varepsilon \chi_\varepsilon\|_{L^\infty} \lesssim_\chi 1$, and $\|\hat{\rho}\|_{L^1} \lesssim_\rho 1$,

$$\|\mathcal{M}(\sigma_\varepsilon)\|_{L^\infty} \lesssim_\rho 1, \quad \|\mathcal{M}(\sigma_\varepsilon) - 1\|_{L^\infty} \lesssim_\rho 1 \quad \text{and} \quad \|\varepsilon \partial_\varepsilon \mathcal{M}(\sigma_\varepsilon)\|_{L^\infty} \lesssim_{\chi, \rho} 1.$$

This implies (A.9) and (A.11) assuming $\varepsilon \langle \xi \rangle \leq 4$, and (A.10) and (A.12) assuming $\varepsilon \langle \xi \rangle \geq 1/2$.

Since $|\langle \xi \rangle - \langle \eta \rangle| \leq |\xi - \eta|$, if $\varepsilon \langle \xi \rangle \geq 4$ and $|\xi - \eta| \leq 2\varepsilon^{-1}$, then

$$\langle \eta \rangle \geq \langle \xi \rangle - |\xi - \eta| \geq \frac{1}{2} \langle \xi \rangle.$$

Therefore and since $\text{supp}(\chi) \subseteq [-2, 2]$ and $\varepsilon \leq 1$, if $\varepsilon \langle \xi \rangle \geq 4$, then

$$|\mathcal{M}(\sigma_\varepsilon)(\xi)| \leq \int_{|\xi - \eta| \leq 2\varepsilon^{-1}} |\hat{\rho}(\eta)| d\eta \lesssim_{\rho, k} \varepsilon^{-1} \langle \xi \rangle^{-k-1} \lesssim_k \varepsilon^{-k} \langle \xi \rangle^{-k}.$$

Combined with the above observation, this proves (A.9) unconditionally. The same argument also proves (A.11) unconditionally.

Since $|\xi - \eta| \leq |\xi| + |\eta| \leq \langle \xi \rangle + \langle \eta \rangle$, if $\varepsilon \langle \xi \rangle \leq 1/2$ and $|\xi - \eta| \geq \varepsilon^{-1}$, then

$$\langle \eta \rangle \geq |\xi - \eta| - \langle \xi \rangle \geq \frac{1}{2} \varepsilon^{-1}.$$

Therefore and since $\text{supp}(\chi - 1) \cap [-1, 1] = \emptyset$ and $\langle \xi \rangle \geq 1$, if $\varepsilon \langle \xi \rangle \leq 1/2$, then

$$|\mathcal{M}(\sigma_\varepsilon)(\xi) - 1| \leq \int_{|\xi - \eta| \geq \varepsilon^{-1}} |\hat{\rho}(\eta)| d\eta \lesssim_{\rho, k} \int_{|\xi - \eta| \geq \varepsilon^{-1}} \langle \eta \rangle^{-k-2} d\eta \lesssim_k \varepsilon^k \lesssim_k \varepsilon^k \langle \xi \rangle^k.$$

Combined with the above observation, this proves (A.10) unconditionally. The same argument also proves (A.12) unconditionally.

(2) By direct computation,

$$\mathcal{M}(\log(r)\sigma_\varepsilon) = \frac{i}{2\pi} \chi'_\varepsilon * \hat{\rho}.$$

Since $\|\chi'_\varepsilon\|_{L^\infty} \lesssim_\chi \varepsilon$ and $\|\varepsilon \partial_\varepsilon \chi'_\varepsilon\|_{L^\infty} \lesssim_\chi \varepsilon$, and $\text{supp}(\chi'_\varepsilon) \subseteq [-2\varepsilon^{-1}, 2\varepsilon^{-1}]$ and $\text{supp}(\chi'_\varepsilon) \cap [-\varepsilon^{-1}, \varepsilon^{-1}] = \emptyset$, the above arguments prove the asserted estimates. $\blacksquare$

Proof of Proposition A.6. Choose χ and ρ as in Lemma A.8 and construct σ_ε accordingly. Moreover, for every $\nu \in \mathbb{R}$ set $\sigma_\varepsilon^{(\nu)} := r^{-\nu} \sigma_\varepsilon$ and $\delta_\varepsilon^{(\nu)} := \log(r) \sigma_\varepsilon^{(\nu)}$. Observe that $\sigma_\varepsilon^{(\nu)}$ arises from χ and $\rho^{(\nu)}$ defined by $\rho^{(\nu)}(x) := e^{-\nu x} \rho(x)$. In particular, Lemma A.8 applies to $\sigma_\varepsilon^{(\nu)}$ mutatis mutandis.

By (A.7) and Lemma A.8 (1), $(S_\varepsilon := \sigma_\varepsilon * \mathcal{M} - : \varepsilon \in (0, 1))$ satisfies the analogues of (A.2), (A.3), (A.4), and (A.5) with respect to $\|-\|_{H_b^k}$.

Therefore, it remains to analyse the behaviour of the five differential operators appearing in $\|-\|_{K_\lambda^k}$ with respect to S_ε . By direct computation,

$$\begin{aligned} r \partial_r (\sigma * \mathcal{M} \phi) &= \sigma * \mathcal{M} r \partial_r \phi, \\ r^\nu (\sigma * \mathcal{M} \phi) &= (r^\nu \sigma) * \mathcal{M} (r^\nu \phi), \quad \text{and} \\ \log(r) (\sigma * \mathcal{M} \phi) &= \sigma * \mathcal{M} \log(r) \phi + \log(r) \sigma * \mathcal{M} \phi. \end{aligned}$$

Consequently,

$$\begin{aligned}
\partial_r S_\varepsilon \phi &= \sigma_\varepsilon^{(1)} *_{\mathcal{M}} \partial_r \phi, \\
\frac{\lambda}{r} S_\varepsilon \phi &= \sigma_\varepsilon^{(1)} *_{\mathcal{M}} \frac{\lambda}{r} \phi, \\
\log(r) (\partial_r - \frac{\lambda}{r}) S_\varepsilon \phi &= \sigma_\varepsilon^{(1)} *_{\mathcal{M}} \log(r) (\partial_r - \frac{\lambda}{r}) \phi + \delta_\varepsilon^{(1)} *_{\mathcal{M}} (\partial_r - \frac{\lambda}{r}) \phi, \\
\partial_r \log(r) (\partial_r - \frac{\lambda}{r}) S_\varepsilon \phi &= \sigma_\varepsilon^{(2)} *_{\mathcal{M}} \partial_r \log(r) (\partial_r - \frac{\lambda}{r}) \phi + \delta_\varepsilon^{(2)} *_{\mathcal{M}} \partial_r (\partial_r - \frac{\lambda}{r}) \phi, \\
\frac{\lambda}{r} \log(r) (\partial_r - \frac{\lambda}{r}) S_\varepsilon \phi &= \sigma_\varepsilon^{(2)} *_{\mathcal{M}} \frac{\lambda}{r} \log(r) (\partial_r - \frac{\lambda}{r}) \phi + \delta_\varepsilon^{(2)} *_{\mathcal{M}} \frac{\lambda}{r} (\partial_r - \frac{\lambda}{r}) \phi.
\end{aligned}$$

This combined with

$$\begin{aligned}
\|(\partial_r - \frac{\lambda}{r}) \phi\|_{H_b^k} &\lesssim \|\log(r) (\partial_r - \frac{\lambda}{r}) \phi\|_{H_b^k}, \\
\|\partial_r (\partial_r - \frac{\lambda}{r}) \phi\|_{H_b^k} &\lesssim \|\partial_r \log(r) (\partial_r - \frac{\lambda}{r}) \phi\|_{H_b^k} + \|\frac{\lambda}{r} (\partial_r - \frac{\lambda}{r}) \phi\|_{H_b^k}, \\
\|\frac{\lambda}{r} (\partial_r - \frac{\lambda}{r}) \phi\|_{H_b^k} &\lesssim \|\frac{\lambda}{r} \log(r) (\partial_r - \frac{\lambda}{r}) \phi\|_{H_b^k},
\end{aligned}$$

and [Lemma A.8](#) proves the assertion. ■

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