A compactness theorem for the Seiberg–Witten equation with multiple spinors in dimension three

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Abstract

We prove that a sequence of solutions of the Seiberg–Witten equation with multiple spinors in dimension three can degenerate only by converging (after rescaling) to a Fueter section of a bundle of moduli spaces of ASD instantons.

Changes to the published version This is an update of our article published as Geometric and Functional Analysis 25 (2015), no. 6, 1799–1821. The present version corrects a mistake in Proposition A.1 pointed out to us by Aleksander Doan, namely the connection A does not need to be flat if $n \geq 3$. This is because the canonical connection on $\mu^{-1}(0) \to \mathring{M}_{1,n}$ is flat only if $n = 2$, whereas we claimed this to be true for all n. This was used to deduce that the limit connection A in Theorem 1.5 is flat with $\mathbf{Z}_2$–monodromy.

1 Introduction

Let M be an oriented Riemannian closed three–manifold. Fix a Spin–structure $\mathfrak{s}$ on M and denote by $\slashed{S}$ the associated spinor bundle; also fix a U(1)–bundle $\mathscr{L}$ over M, a positive integer $n \in \mathbf{N}$ and a SU(n)–bundle E together with a connection B. We consider pairs $(A, \Psi) \in \mathscr{A}(\mathscr{L}) \times \Gamma\left(\mathrm{Hom}(E, \slashed{S} \otimes \mathscr{L})\right)$ consisting of a connection A on $\mathscr{L}$ and an n–tuple of twisted spinors Ψ satisfying the *Seiberg–Witten equation with n spinors*:

$$\begin{aligned} \slashed{D}_{A\otimes B}\Psi &= 0 \quad \text{and} \\ F_A &= \mu(\Psi). \end{aligned} \tag{1.1}$$

Here $\mu\colon \mathrm{Hom}(E, \slashed{S} \otimes \mathscr{L}) \to \mathfrak{g}_{\mathscr{L}} \otimes \mathfrak{su}(\slashed{S}) = i\mathfrak{su}(\slashed{S})$ is defined by

$$\mu(\Psi) \coloneqq \Psi\Psi^* - \frac{1}{2}|\Psi|^2 \,\mathrm{id}_{\slashed{S}} \tag{1.2}$$

and we identify $\Lambda^2 T^*M$ with $\mathfrak{su}(\slashed{S})$ via

$$e^i \wedge e^j \mapsto \frac{1}{2}[\gamma(e^i), \gamma(e^j)] = \varepsilon_{ijk}\gamma(e^k). \tag{1.3}$$

If $n = 1$, then E and B are trivial, since $\mathrm{SU}(1) = \{1\}$, and (1.1) is nothing but the classical Seiberg–Witten equation in dimension three, which has been studied with remarkable success, see, e.g., [Che97; Lim00; KM07]. A key ingredient in the analysis of (1.1) with $n = 1$ is the identity

$$\langle \mu(\Psi)\Psi, \Psi \rangle = \frac{1}{2}|\Psi|^4,$$

which combined with the Weitzenböck formula yields an a priori bound on Ψ and, therefore, immediately gives compactness of the moduli spaces of solutions to (1.1). After taking care of issues to do with transversality and reducibles, counting solutions of (1.1) leads to an invariant of three–manifolds.

The above identity does not hold for $n \geqslant 2$ and, more importantly, μ is no longer proper; hence, the L^2–norm of Ψ is not bounded a priori. From an analytical perspective the difficult case is when this L^2–norm becomes very large; however, also the case of very small L^2–norm deserves special attention as it corresponds to reducible solutions of (1.1). With this in mind it is natural to blow-up (1.1), that is, to consider triples $(A, \Psi, \alpha) \in \mathscr{A}(\mathscr{L}) \times \Gamma\left(\mathrm{Hom}(E, \$ \otimes \mathscr{L})\right) \times [0, \pi/2]$ satisfying

$$\begin{aligned} \|\Psi\|_{L^2} &= 1, \\ \not{D}_{A\otimes B}\Psi &= 0 \quad \text{and} \\ \sin(\alpha)^2 F_A &= \cos(\alpha)^2 \mu(\Psi), \end{aligned} \tag{1.4}$$

c.f. [KM07, Section 2.5]. The difficulty in the analysis can now be understood as follows: for $\alpha \in (0, \pi/2]$ equation (1.4) is elliptic (after gauge fixing), but as α tends to zero it becomes degenerate.

The following is the main result of this article:

Theorem 1.5. *Let $(A_i, \Psi_i, \alpha_i) \in \mathscr{A}(\mathscr{L}) \times \Gamma\left(\mathrm{Hom}(E, \$ \otimes \mathscr{L})\right) \times (0, \pi/2]$ be a sequence of solutions of (1.4). If $\limsup \alpha_i > 0$, then after passing to a subsequence and up to gauge transformations (A_i, Ψ_i, α_i) converges smoothly to a limit (A, Ψ, α). If $\limsup \alpha_i = 0$, then after passing to a subsequence the following holds:*

- *There is a closed nowhere-dense subset $Z \subset M$, a connection A on $\mathscr{L}|_{M\setminus Z}$ and a spinor $\Psi \in \Gamma\left(M\setminus Z, \mathrm{Hom}(E, \$ \otimes \mathscr{L})\right)$ such that $(A, \Psi, 0)$ solves (1.4). $|\Psi|$ extends to a Hölder continuous function on all of M and $Z = |\Psi|^{-1}(0)$.*
- *On $M\setminus Z$, up to gauge transformations, A_i converges weakly in $W^{1,2}_{\mathrm{loc}}$ to A and Ψ_i converges weakly in $W^{2,2}_{\mathrm{loc}}$ to Ψ. There is a constant $\gamma > 0$ such that $|\Psi_i|$ converges to $|\Psi|$ in $C^{0,\gamma}$ on all of M.*

Remark 1.6. Proposition A.3 gives more detailed information about the limit $(A, \Psi, 0)$. In particular, if $n = 2$, then A is flat with monodromy in $\mathbf{Z}_2$.

Remark 1.7. Theorem 1.5 should be compared with the results of Taubes on $\mathbf{PSL}(2, \mathbf{C})$–connections on three–manifolds with curvature bounded in L^2 [Tau13, Theorem 1.1]. Our proof heavily relies on his insights and techniques.

Remark 1.8. Taubes' very recent work [Tau14, Theorems 1.2, 1.3, 1, 4 and 1.5] implies detailed regularity properties for Z; in particular, Z has Hausdorff dimension at most one. To see that his theorems apply in our situation note that Z is the zero locus of a $\mathbf{Z}_2$ harmonic spinor by Appendix A.

As is discussed in Appendix A, gauge equivalence classes of nowhere-vanishing solutions of (1.4) with $\alpha = 0$ correspond to Fueter sections of a bundle $\mathfrak{M}$ with fibre $\mathring{M}_{1,n}$, the framed moduli space of centred charge one $\mathrm{SU}(n)$ ASD instantons on $\mathbf{R}^4$. In particular, while (1.4) degenerates as α tends to zero, for $\alpha = 0$ it is equivalent to an elliptic partial differential equation, away from the zero-locus of Ψ. Morally, this is why one can hope to prove Theorem 1.5.

In view of Theorem 1.5, the count of solutions of (1.4) can depend on the choice of (generic) parameters in $\mathscr{P}$ (the space of metrics on M and connections on E): since $\mathring{M}_{1,n}$ is a cone and the Fueter equation has index zero, one expects Fueter sections of $\mathfrak{M}$ to appear (only) in codimension one; thus, the count of solutions of (1.4) can jump along a path of parameters in $\mathscr{P}$. In other words: there is a set $\mathscr{W} \subset \mathscr{P}$ of codimension one and the number of solutions of (1.4) depends on the connected component of $\mathscr{P}\backslash\mathscr{W}$. In the study of gauge theory on G_2–manifolds the count of G_2–instantons also undergoes a jump whenever a solution of (1.4) with $\alpha = 0$ appears, with M an associative submanifold of a G_2–manifold and B the restriction of a G_2–instanton to M, see [DS11; Wal17; Wal13]. So while both the count of G_2–instantons and the count of solutions of (1.4) cannot be invariants, there is hope that a suitable combination of counts of G_2–instantons and solutions of (generalisations of) (1.4) on associative submanifolds will yield an invariant of G_2–manifolds. We will discuss this circle of ideas in more detail elsewhere.

Outline of the proof of Theorem 1.5 The Weitzenböck formula leads to a priori bounds which directly prove the first half of Theorem 1.5. The proof of the second half is more involved. For a solution (A, Ψ, α) of (1.4), we show that the (renormalised) $W^{2,2}_A$–norm of Ψ on a ball $B_r(x)$ is uniformly bounded provided the radius is smaller than the *critical radius*

$$\rho = \sup\{r : r^{1/2}\|F_A\|_{L^2(B_r(x))} \leqslant 1\}.$$

To control ρ we use a *frequency function* $\mathrm{N}(r)$, which—roughly speaking—measures the vanishing order of Ψ near x. More precisely, we prove that there exists a constant $\omega > 0$, depending only on the geometry of M, such that $\mathrm{N}(50r) \leqslant \omega$ implies $\rho \geq r$. We also show that for any $\omega, \varepsilon > 0$ there exists $r > 0$ such that $\mathrm{N}(r) \leqslant \omega$ provided $|\Psi|(x) \geqslant \varepsilon$. Thus, we can establish convergence outside the subset $Z = \{x \in M : \limsup |\Psi_i|(x) = 0\}$.

Convention 1.9. We write $x \lesssim y$ (or $y \gtrsim x$) for $x \leqslant cy$ with $c > 0$ a universal constant, which depends only on the geometry of M, E and B; should c depend on further data we indicate that by a subscript. $O(x)$ denotes a quantity y with $|y| \lesssim x$. We denote by r_0 a constant $0 < r_0 \ll 1$; in particular, $r_0 \leqslant \mathrm{injrad}(M)$. We assume that all radii r on M under consideration are less than r_0. Throughout the rest of this article $\mathscr{L}$, E and B are fixed.

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2 A priori estimates

In this section we prove the following a priori estimates:

Proposition 2.1. *Every solution* $(A, \Psi, \alpha) \in \mathscr{A}(\mathscr{L}) \times \Gamma\left(\mathrm{Hom}(E, \$ \otimes \mathscr{L})\right) \times (0, \pi/2]$ *of* (1.4) *satisfies*

$$\|\Psi\|_{L^\infty} = O(1)$$

and, for each $x \in M$ *and* $r > 0$*,*

$$\begin{aligned} \|\nabla_{A\otimes B}\Psi\|_{L^2(B_r(x))} &= O(r^{1/2}) \quad \text{and} \\ \|\mu(\Psi)\|_{L^2(B_r(x))} &= O(r^{1/2}\tan(\alpha)). \end{aligned}$$

This implies the first part of Theorem 1.5 because if $\limsup \alpha_i > 0$, then (1.4) does not degenerate and standard methods apply:

Proposition 2.2. *In the situation of Theorem 1.5 if* $\limsup \alpha_i > 0$*, then, after passing to a subsequence and up to gauge transformations,* (A_i, Ψ_i, α_i) *converges in* C^∞ *to a limit* (A, Ψ, α) *solving* (1.4).

By the Banach–Alaoglu theorem we have the following proposition:

Proposition 2.3. *In the situation of Theorem 1.5 after passing to a subsequence* $|\Psi_i|$ *converges weakly in* $W^{1,2}$ *to a bounded limit* $|\Psi|$*.*

Remark 2.4. Note that we have not yet constructed Ψ; however, we will show later that the notation $|\Psi|$ is indeed justified.

The key to proving Proposition 2.1 are the Weitzenböck formula (2.6), the algebraic identity (2.8) and the integration by parts formula (2.11).

Proposition 2.5. *For all* $(A, \Psi) \in \mathscr{A}(\mathscr{L}) \times \Gamma\left(\mathrm{Hom}(E, \$ \otimes \mathscr{L})\right)$

$$\not{D}^*_{A\otimes B}\not{D}_{A\otimes B}\Psi = \nabla^*_{A\otimes B}\nabla_{A\otimes B}\Psi + \frac{s}{4}\Psi + F_A\Psi + F_B\Psi \tag{2.6}$$

with s *denoting the scalar curvature of* g *and* F_A *and* F_B *acting via the isomorphism defined in* (1.3). □

Proposition 2.7. *For all* $\Psi \in \Gamma\left(\mathrm{Hom}(E, \slashed{S} \otimes \mathscr{L})\right)$

$$\langle \mu(\Psi)\Psi, \Psi\rangle = |\mu(\Psi)|^2. \tag{2.8}$$

Proof. This follows from a simple computation:

$$\langle \mu(\Psi)\Psi, \Psi\rangle = \langle \mu(\Psi), \Psi\Psi^*\rangle = \langle \mu(\Psi), \Psi\Psi^* - \frac{1}{2}|\Psi|^2 \,\mathrm{id}_{\slashed{S}}\rangle = |\mu(\Psi)|^2. \qquad \square$$

Proposition 2.9. *Suppose* $(A, \Psi, \alpha) \in \mathscr{A}(\mathscr{L}) \times \Gamma\left(\mathrm{Hom}(E, \slashed{S} \otimes \mathscr{L})\right) \times (0, \pi/2]$ *satisfies*

$$\begin{aligned} \slashed{D}_{A\otimes B}\Psi &= 0 \quad \textit{and} \\ \sin(\alpha)^2 F_A &= \cos(\alpha)^2 \mu(\Psi). \end{aligned} \tag{2.10}$$

If f is any smooth function on M and U is a closed subset of M with smooth boundary, then

$$\begin{aligned} \int_U \Delta f \cdot |\Psi|^2 + f \cdot \left(\frac{s}{2}|\Psi|^2 + 2\left\langle F_B\Psi, \Psi\right\rangle + 2\tan(\alpha)^{-2}|\mu(\Psi)|^2 + 2|\nabla_A\Psi|^2\right) \\ = \int_{\partial U} f \cdot \partial_\nu |\Psi|^2 - \partial_\nu f \cdot |\Psi|^2. \end{aligned} \tag{2.11}$$

Here ν denotes the outward pointing normal vector field.

Proof. Combine (1.4), (2.6) and (2.8) to obtain

$$\frac{1}{2}\Delta|\Psi|^2 + \frac{s}{4}|\Psi|^2 + \langle F_B\Psi, \Psi\rangle + \tan(\alpha)^{-2}|\mu(\Psi)|^2 + |\nabla_A\Psi|^2 = 0. \tag{2.12}$$

The identity (2.11) now follows from

$$\int_U \Delta f \cdot g - f \cdot \Delta g = \int_{\partial U} f \cdot \partial_\nu g - \partial_\nu f \cdot g$$

with $g = |\Psi|^2$. $\square$

Proof of Proposition 2.1. Apply Proposition 2.9 with $f = 1$ and $U = M$ to obtain

$$\int_M |\nabla_A\Psi|^2 \leqslant -\int_M \frac{s}{4}|\Psi|^2 + \langle F_B\Psi, \Psi\rangle = O(1).$$

Combine this with Kato's inequality and the Sobolev embedding $W^{1,2} \hookrightarrow L^6$ to obtain

$$\|\Psi\|_{L^6} = O(1).$$

The operator $\Delta + 1$ is invertible and has a positive Green's function G, which has an expansion of the form

$$G(x, y) = \frac{1}{4\pi}\frac{e^{-d(x,y)}}{d(x, y)} + O\left(d(x, y)\right).$$

Apply Proposition 2.9 with $f = G(x, \cdot)$ and $U = M \backslash B_\sigma(x)$, and pass to the limit $\sigma = 0$ to obtain

$$\frac{1}{2}|\Psi|^2(x) + \int_M G(x,\cdot)\left(\tan(\alpha)^{-2}|\mu(\Psi)|^2 + |\nabla_A\Psi|^2\right) \lesssim \int_M G(x,\cdot)|\Psi|^2.$$

The right-hand side of this equation is $O(1)$ because of the L^6–bound on Ψ. Taking the supremum of the left-hand side over all $x \in M$ yields the desired bounds. $\square$

3 Curvature controls Ψ

This section begins the proof of the more difficult second part of Theorem 1.5.

Definition 3.1. The *critical radius* $\rho(x)$ of a connection $A \in \mathscr{A}(\mathscr{L})$ is defined by

$$\rho(x) \coloneqq \sup\{r \in (0, r_0] : r^{1/2}\|F_A\|_{L^2(B_r(x))} \leqslant 1\}.$$

If the base-point x is obvious from the context and confusion is unlikely to arise, we will often drop x from the notation and just write ρ.

Remark 3.2. While some constant must be chosen in the definition of ρ, the precise choice is immaterial, since we are working with an abelian gauge group $G = \mathrm{U}(1)$. In general, 1 should be replaced by the Uhlenbeck constant of G on M.

Proposition 3.3. *Suppose* $(A, \Psi) \in \mathscr{A}(\mathscr{L}) \times \Gamma\left(\mathrm{Hom}(E, \$ \otimes \mathscr{L})\right)$ *satisfies*

$$\not{D}_{A\otimes B}\Psi = 0.$$

If $x \in M$ *and* $\delta \in (0, 1]$*, then*

$$r^{1/2}\|\nabla^2_{A\otimes B}\Psi\|_{L^2(B_{(1-\delta)r}(x))} \leqslant f_\delta\left(\|\Psi\|_{L^\infty(B_r(x))}, r^{-1/2}\|\nabla_A\Psi\|_{L^2(B_r(x))}, r^{1/2}\|F_A\|_{L^2(B_r(x))}\right).$$

Here f_δ *is monotone increasing in all of its arguments. In particular, if* $(A, \Psi, \alpha) \in \mathscr{A}(\mathscr{L}) \times \Gamma\left(\mathrm{Hom}(E, \$ \otimes \mathscr{L})\right) \times [0, \pi/2]$ *is a solution of* (1.4)*, then*

$$\rho^{1/2}\|\nabla^2_{A\otimes B}\Psi\|_{L^2(B_{\rho/2}(x))} = O(1).$$

Proof. The statement is scale-invariant, so we might as well assume that $B_r(x)$ is a geodesic ball B_1 of radius one (with an almost flat metric). Fix a cut-off function χ which is supported in $B_{1-\delta/2}$ and is equal to one in $B_{1-\delta}$. A straight-forward direct computation using integration by parts yields

$$\begin{aligned}\int |\nabla^2_{A\otimes B}(\chi\Psi)|^2 \lesssim \int &|\nabla^*_{A\otimes B}\nabla_{A\otimes B}(\chi\Psi)|^2 \\ &+ |F_{A\otimes B}||\nabla_{A\otimes B}(\chi\Psi)|^2 + |F_{A\otimes B}||\chi\Psi||\nabla^2_{A\otimes B}(\chi\Psi)|.\end{aligned}$$

Since, as a consequence of the Weitzenböck formula (2.6),

$$\nabla^*_{A\otimes B}\nabla_{A\otimes B}(\chi\Psi) = -\frac{s}{4}\chi\Psi - F_{A\otimes B}(\chi\Psi) - 2\nabla^{A\otimes B}_{\nabla\chi}\Psi + (\Delta\chi)\Psi,$$

we can write

$$\begin{aligned} \int |\nabla^2_{A\otimes B}(\chi\Psi)|^2 \lesssim_\delta \int & |F_{A\otimes B}|^2|\chi\Psi|^2 + |F_{A\otimes B}||\nabla_{A\otimes B}(\chi\Psi)|^2 \\ & + |F_{A\otimes B}||\chi\Psi||\nabla^2_{A\otimes B}(\chi\Psi)| \\ & + |\nabla_{A\otimes B}\Psi|^2 + |\Psi|^2. \end{aligned} \tag{3.4}$$

The first and the last two terms are already acceptable. The third term is bounded by

$$\varepsilon^{-1}\|F_{A\otimes B}\|^2_{L^2}\|\Psi\|^2_{L^\infty} + \varepsilon\|\nabla^2_{A\otimes B}(\chi\Psi)\|^2_{L^2}$$

for all $\varepsilon > 0$. The first term is acceptable and the second one can be rearranged to the left-hand side of (3.4) provided ε is chosen sufficiently small. The second term can be bounded by

$$\|F_{A\otimes B}\|_{L^2}\|\nabla_{A\otimes B}(\chi\Psi)\|^2_{L^4}.$$

Using the Gagliardo–Nirenberg interpolation inequality

$$\|f\|_{L^4} \lesssim \|\nabla f\|^{3/4}_{L^2}\|f\|^{1/4}_{L^2}$$

and Kato's inequality we obtain

$$\begin{aligned} \|\nabla_{A\otimes B}(\chi\Psi)\|^2_{L^4} &\lesssim \|\nabla^2_{A\otimes B}(\chi\Psi)\|^{3/2}_{L^2}\|\nabla_{A\otimes B}(\chi\Psi)\|^{1/2}_{L^2} \\ &\leqslant \varepsilon\|\nabla^2_{A\otimes B}(\chi\Psi)\|^2_{L^2} + \varepsilon^{-3}\|\nabla_{A\otimes B}(\chi\Psi)\|^2_{L^2} \end{aligned}$$

for all $\varepsilon > 0$. The first term can be rearranged to the left-hand side of (3.4) provided ε is chosen sufficiently small and the second term is acceptable. □

4 A frequency function

In view of Proposition 3.3 the following result is the key to proving Theorem 1.5.

Proposition 4.1. *There exists a constant $\omega > 0$ such that for each solution $(A, \Psi, \alpha) \in \mathscr{A}(\mathscr{L}) \times \Gamma\left(\mathrm{Hom}(E, \slashed{S} \otimes \mathscr{L})\right) \times (0, \pi/2)$ of (1.4) we have*

$$\rho(x) \gtrsim \min\{1, |\Psi|^{1/\omega}(x)\}.$$

The proof of this proposition will be given in Section 5. In this section we lay the groundwork by introducing the following tool:

Definition 4.2. Given $(A, \Psi, \alpha) \in \mathscr{A}(\mathscr{L}) \times \Gamma\left(\mathrm{Hom}(E, \$ \otimes \mathscr{L})\right) \times (0, \pi/2)$, the *frequency function* $\textsc{n}_x \colon (0, r_0] \to [0, \infty)$ at $x \in M$ is defined by

$$\textsc{n}_x(r) \coloneqq \frac{rH_x(r)}{h_x(r)}$$

with

$$H_x(r) \coloneqq \int_{B_r(x)} |\nabla_{A\otimes B}\Psi|^2 + \tan(\alpha)^{-2}|\mu(\Psi)|^2$$
$$\text{and} \quad h_x(r) \coloneqq \int_{\partial B_r(x)} |\Psi|^2.$$

If the base-point x is obvious from the context and confusion is unlikely to arise, we will often drop x from the notation and just write $\textsc{n}$, H and h.

Remark 4.3. The notion of frequency function, introduced by Almgren [Alm79], is important in the study of singular/critical sets of elliptic partial differential equations, see, e.g., [HHL98; NV14]. Our frequency function is an adaptation of the one used by Taubes in [Tau13].

Throughout the rest of this section we will assume that $(A, \Psi, \alpha) \in \mathscr{A}(\mathscr{L}) \times \Gamma\left(\mathrm{Hom}(E, \$ \otimes \mathscr{L})\right) \times (0, \pi/2)$ satisfies (1.4) and fix a point $x \in M$. We establish various important properties of the frequency function. In particular, we show that:

- $\textsc{n}$ is almost monotone increasing in r.
- $\textsc{n}$ controls the growth of h.
- If $|\Psi|(x) > 0$, then $\textsc{n}(r)$ goes to zero as r goes to zero.

Moreover, we study the base-point dependence of $\textsc{n}$.

4.1 Almost monotonocity of N

Proposition 4.4. *The derivative of the frequency is bounded below as follows*

$$\textsc{n}'(r) \geqslant O(r)(1 + \textsc{n}(r)). \tag{4.5}$$

Before we embark on the proof, which occupies the remainder of this subsection, let us note the following consequence:

Proposition 4.6. *If* $0 < s \leqslant r$, *then*

$$\textsc{n}(s) \leqslant e^{O\left(r^2 - s^2\right)}\textsc{n}(r) + O\left(r^2 - s^2\right). \tag{4.7}$$

Proof. From (4.5) it follows that

$$\partial_r \log(\mathrm{N}(r) + 1) \geqslant -2cr.$$

This integrates to

$$\log(\mathrm{N}(r) + 1) - \log(\mathrm{N}(s) + 1) \geqslant -c(r^2 - s^2),$$

i.e.,

$$\mathrm{N}(s) + 1 \leqslant e^{c(r^2 - s^2)}(\mathrm{N}(r) + 1),$$

which directly implies (4.7). □

The derivative of the frequency is

$$\mathrm{N}'(r) = \frac{H(r)}{h(r)} + \frac{rH'(r)}{h(r)} - \frac{rh'(r)H(r)}{h(r)^2}; \tag{4.8}$$

hence, to prove Proposition 4.4 we need to better understand h' and H'. This is what is achieved in the following.

Proposition 4.9. *The derivative of h satisfies*

$$h'(r) = 2h(r)/r + \int_{\partial B_r(x)} \partial_r |\Psi|^2 + O(r)h(r) \tag{4.10}$$

and

$$h'(r) = \left(2 + 2\mathrm{N}(r) + O(r^2)\right) h(r)/r. \tag{4.11}$$

Moreover,

$$\int_{B_r(x)} |\Psi|^2 \lesssim rh(r). \tag{4.12}$$

Proof. We proceed in four steps.

Step 1. *The identity (4.10) is clear if the metric is flat near x; the term $O(r)h(r)$ compensates for the metric possibly being non-flat.*

Step 2. $\int_{B_r(x)} |\Psi|^2 \lesssim (1 + \mathrm{N}(r))rh(r)$.

Apply the following general fact

$$\int_{B_r(x)} \mathrm{d}(x, \cdot)^{-2} f^2 \lesssim r^{-1} \int_{\partial B_r(x)} f^2 + \int_{B_r(x)} |\mathrm{d}f|^2,$$

which can be proved using integration by parts and Cauchy–Schwarz, to $f = |\Psi|$ and use Kato's inequality.

Step 3. $h'(r) > 0$.

Use Proposition 2.9 with $U = B_r(x)$ and $f = 1$ to write

$$\int_{\partial B_r(x)} \partial_r |\Psi|^2 = 2H(r) + O(1) \int_{B_r(x)} |\Psi|^2. \tag{4.13}$$

The estimate from Step 2 implies

$$h'(r) = \left(1 + O(r^2)\right)(2 + 2\textsc{n}(r))h(r)/r$$

which is non-negative because $r \leqslant r_0$.

Step 4. *Proof of* (4.11) *and* (4.12).

The bound (4.12) follows directly from $h'(r) > 0$. Using (4.12) in Step 3 instead of the estimate from Step 2 immediately implies (4.11). □

Proposition 4.14. *The derivative of H satisfies*

$$H'(r) = \frac{1}{r}H(r) + \int_{\partial B_r(x)} 2|\nabla_r^{A\otimes B}\Psi|^2 + \tan(\alpha)^{-2}|i(\partial_r)\mu(\Psi)|^2 + O\big((1 + \textsc{n}(r))h(r)\big).$$

Here we think of $\mu(\Psi)$ as a 2–form via (1.3).

Proof. The punctured ball $\dot{B}_{r_0}(x) \coloneqq B_{r_0}(x)\backslash\{x\}$ is foliated by the surfaces $\partial B_r(x)$ with normal vector field ∂_r. According to [BGM05, Section 3] the restriction of the spin bundle on $\dot{B}_{r_0}(x)$ to $\partial B_r(x)$ can be identified with the spin bundle on $\partial B_r(x)$ and if $\tilde{\gamma}$, $\tilde{\nabla}$ and $\tilde{\slashed{D}}$ denote the Clifford multiplication, spin connection and Dirac operator on $\partial B_r(x)$ respectively, then for $v \in T\partial B_r(x)$:

$$\begin{aligned} \gamma(v) &= -\gamma(\partial_r)\tilde{\gamma}(v), \\ \nabla_v &= \tilde{\nabla}_v + \frac{e^{O(r^2)}}{2r}\tilde{\gamma}(v) \quad \text{and} \\ \slashed{D} &= \gamma(\partial_r)(\nabla_r + \frac{e^{O(r^2)}}{r} - \tilde{\slashed{D}}). \end{aligned}$$

(If the metric on $B_{r_0}(x)$ is flat, then the mean curvature of $\partial B_r(x)$ is $-\frac{1}{r}$. In general, there is a correction term; hence, the term $e^{O(r^2)}$.) In particular, $\slashed{D}\Psi = 0$ is equivalent to

$$\tilde{\slashed{D}}\Psi = \nabla_r\Psi + \frac{e^{O(r^2)}}{r}\Psi.$$

For Ψ a harmonic spinor on $B_r(x)$ we compute:

$$\begin{aligned}\int_{\partial B_r(x)} |\nabla\Psi|^2 - |\nabla_r\Psi|^2 &= \int_{\partial B_r(x)} |\tilde{\nabla}\Psi + \frac{e^{O(r^2)}}{2r}\tilde{\gamma}(\cdot)\Psi|^2 \\ &= \int_{\partial B_r(x)} |\tilde{\nabla}\Psi|^2 - \frac{e^{O(r^2)}}{r}\langle \tilde{\not{D}}\Psi, \Psi\rangle + \frac{e^{O(r^2)}}{2r^2}|\Psi|^2 \\ &= \int_{\partial B_r(x)} |\tilde{\nabla}\Psi|^2 - \frac{e^{O(r^2)}}{r}\langle \nabla_r\Psi, \Psi\rangle - \frac{e^{O(r^2)}}{2r^2}|\Psi|^2.\end{aligned}$$

Using the Weitzenböck formula (2.6) the first term can be written as

$$\begin{aligned}\int_{\partial B_r(x)} |\tilde{\nabla}\Psi|^2 &= \int_{\partial B_r(x)} \langle \tilde{\nabla}^*\tilde{\nabla}\Psi, \Psi\rangle \\ &= \int_{\partial B_r(x)} |\tilde{\not{D}}\Psi|^2 - \frac{e^{O(r^2)}}{2r^2}|\Psi|^2 \\ &= \int_{\partial B_r(x)} |\nabla_r\Psi|^2 + \frac{2e^{O(r^2)}}{r}\langle \nabla_r\Psi, \Psi\rangle + \frac{e^{O(r^2)}}{2r^2}|\Psi|^2.\end{aligned}$$

This combined with (4.13) and (4.12) proves the asserted identity if A and B are product connections.

If A and B are not the product connection, the computation is identical up to changes in notation *and* in the Weitzenböck formula two additional terms appear. The first is

$$-\int_{\partial B_r(x)} \left\langle F_A|_{\partial B_r(x)}, \mu(\Psi)\right\rangle$$

and the second can be estimated by $O(1)h(r)$. If (e_1, e_2) is a local positive orthonormal frame of $T\partial B_r(x)$, then the integrand in the above expression is

$$\frac{1}{2}\langle F_A(e_1, e_2)[\tilde{\gamma}(e_1), \tilde{\gamma}(e_2)], \mu(\Psi)\rangle = \langle F_A(e_1, e_2)\gamma(\partial_r), \mu(\Psi)\rangle .$$

To better understand this term, observe that if $\{\cdot,\cdot\}$ denotes the anti-commutator, then

$$\mu(\Psi) = \sum_m \frac{1}{2}\{\mu(\Psi), \gamma_m\}\gamma_m$$

and $\langle \gamma_m, \gamma_n\rangle = 2\delta_{mn}$. Using $F_A = \tan(\alpha)^{-2}\mu(\Psi)$ we can write the integrand as $\tan(\alpha)^{-2}$ times

$$\frac{1}{2}|\{\mu(\Psi), \gamma(\partial_r)\}|^2 = |\mu(\Psi)|^2 - |i(\partial_r)\mu(\Psi)|^2.$$

This proves (4.14) in general. $\square$

Proof of Proposition 4.4. Plug (4.11) and (4.14) into (4.8) and use (4.13) and (4.12) to obtain

$$
\begin{aligned}
\mathrm{N}(r)' = {} & \frac{2r}{h(r)} \int_{\partial B_r(x)} |\nabla_r^{A\otimes B}\Psi|^2 + \tan(\alpha)^{-2}|i(\partial_r)\mu(\Psi)|^2 \\
& - \frac{2r}{h(r)^2} \left(\int_{\partial B_r(x)} \left\langle \nabla_r^{A\otimes B}\Psi, \Psi \right\rangle \right)^2 \\
& + O(r)\,(1 + \mathrm{N}(r))\,.
\end{aligned}
$$

By Cauchy–Schwarz the sum of the first and the third term is positive. This completes the proof. □

4.2 N controls the growth of h

Proposition 4.15. *If $0 < s < r$, then*

$$
h(r) = e^{O(r^2)}\,(r/s)^2 \exp\left(2\int_s^r \mathrm{N}(t)/t\,\mathrm{d}t\right) h(s). \tag{4.16}
$$

Proof. (4.11) can be written as

$$
(\log h(r))' = (2 + 2\mathrm{N}(r))/r + O(r).
$$

Integrating this yields (4.16). □

Corollary 4.17. *If $0 < s < r$, then*

$$
h(s) \lesssim (s/r)^2 h(r).
$$

In particular, if $h(s)$ is positive, then so is $h(r)$; moreover, $|\Psi|^2(x) \lesssim h(r)/r^2$.

Proposition 4.18. *If $0 < s < r$, then*

$$
e^{O(r^2)}(s/r)^{e^{O(r^2)}(2+2\mathrm{N}(r))}h(r) \leqslant h(s) \leqslant e^{O(r^2)}(s/r)^{e^{O(r^2)}(2+2\mathrm{N}(s))}h(r).
$$

Proof. Combine

$$
h(s) = e^{O(r^2)}\,(s/r)^2 \exp\left(-2\int_s^r \mathrm{N}(t)/t\,\mathrm{d}t\right) h(r)
$$

with

$$
\begin{aligned}
\int_s^r \mathrm{N}(t)/t\,\mathrm{d}t &\leqslant \int_s^r \frac{1}{t}\left(e^{O(r^2-t^2)}\mathrm{N}(r) + O(r^2 - t^2)\right)\,\mathrm{d}t \\
&\leqslant -\left(e^{O(r^2)}\mathrm{N}(r) + O(r^2)\right)\log(s/r)
\end{aligned}
$$

and

$$
-\left(e^{O(r^2)}\mathrm{N}(s) + O(r^2)\right)\log(s/r) \leqslant \int_s^r \mathrm{N}(t)/t\,\mathrm{d}t.
$$

The last two inequalities are consequences of Proposition 4.6. □

4.3 $|\Psi|(x)$ controls N

Proposition 4.19. *If* $0 < \omega \ll 1$ *and*

$$s \lesssim_\omega \min\{1, |\Psi|^{1/\omega}(x)\},$$

then $\mathrm{N}(s) \lesssim \omega$.

Proof. By Proposition 2.1, $h(r) \lesssim r^2$ and, by Corollary 4.17, $h_x(s) \gtrsim s^2|\Psi|^2(x)$. From Proposition 4.18 it follows that for $s < r$

$$(r/s)^{e^{O(r^2)}2\mathrm{N}(s)+O(r^2)} \leqslant c^2|\Psi|^{-2}(x);$$

hence,

$$\mathrm{N}(s) \lesssim \frac{\log(c|\Psi|^{-1}(x))}{\log(r/s)} + O(r^2).$$

If $\sigma := c|\Psi|^{-1}(x) \leqslant 1$, then the first term is non-positive and setting $r = 2\omega$ and $s = \omega$ yields the asserted bound. If $\sigma > 1$, set $r = \omega$ and $s = \omega c^{-1/\omega}|\Psi|^{1/\omega}(x) = \omega\sigma^{-1/\omega}$ to obtain

$$\mathrm{N}(s) \lesssim \omega + O(r^2) \lesssim \omega. \qquad \square$$

4.4 Dependence of N on the base-point

Proposition 4.20. *For* $x, y \in M$ *and* $r > 0$

$$h_x(r) \lesssim \frac{2r + d(x,y)}{r} h_y\big(2r + d(x,y)\big).$$

Proof. By Corollary 4.17 and (4.12)

$$rh_x(r) \lesssim \int_{B_{2r}(x)} |\Psi|^2 \leqslant \int_{B_{2r+d(x,y)}(y)} |\Psi|^2 \lesssim \big(2r + d(x,y)\big)h_y\big(2r + d(x,y)\big). \qquad \square$$

Proposition 4.21. *Suppose* $x \in M$ *and* $r > 0$ *are such that* $\mathrm{N}_x(10r) \leqslant 1$*. If* $y \in B_r(x)$*, then* $\mathrm{N}_y(5r) \lesssim \mathrm{N}_x(10r)$*.*

Proof. Since

$$\mathrm{N}_y(5r) = \frac{5rH_y(5r)}{h_y(5r)} \lesssim \mathrm{N}_x(10r)\frac{h_x(10r)}{h_y(5r)},$$

it is key to control the latter quotient. Using Proposition 4.18 with $\mathrm{N}_x(10r) \leqslant 1$ as well as Proposition 4.20

$$h_x(10r) \lesssim h_x(r) \lesssim h_y(5r). \qquad \square$$

5 $\textsc{n}$ controls $\rho(x)$

In view of Proposition 4.19 it suffices to prove the following in order to complete the proof of Proposition 4.1.

Proposition 5.1. *There are $\omega, \rho_0 > 0$ such that for every solution $(A, \Psi, \alpha) \in \mathscr{A}(\mathscr{L}_0) \times \Gamma\left(\mathrm{Hom}(E_0, \$ \otimes \mathscr{L})\right) \times (0, \pi/2)$ of (1.4)*

$$\text{if} \quad \textsc{n}(50r) \leqslant \omega, \quad \text{then} \quad \rho \geq \min\{r, \rho_0\}.$$

5.1 Interior L^2–bounds on the curvature

We first show that if the critical radius ρ and the frequency $\textsc{n}(\rho)$ are very small, then so is the renormalised L^2–norm of F_A on $B_{\rho/2}(x)$:

Proposition 5.2. *Let $(A, \Psi, \alpha) \in \mathscr{A}(\mathscr{L}_0) \times \Gamma\left(\mathrm{Hom}(E_0, \$ \otimes \mathscr{L})\right) \times (0, \pi/2)$ be a solution of (1.4). For any $\varepsilon > 0$, if*

$$\rho \ll_\varepsilon 1 \quad \text{and} \quad \textsc{n}(\rho) \ll_\varepsilon 1,$$

then

$$\rho \int_{B_{\rho/2}(x)} |F_A|^2 \leqslant \varepsilon.$$

Since

$$\frac{\tan(\alpha)^2}{h(\rho)} \leqslant \left(\rho \int_{B_\rho(x)} |F_A|^2\right)^{-1} \textsc{n}(\rho) = \textsc{n}(\rho),$$

this is a direct consequence of the following.

Proposition 5.3. *Denote by (B_r, g) a Riemannian 3–ball of radius $r > 0$, by $\mathscr{L}_0$ a $\mathrm{U}(1)$–bundle over B_r, by E_0 an $\mathrm{SU}(n)$–bundle over B_r and by B a connection on E_0. Suppose that $(A, \Psi, \alpha) \in \mathscr{A}(\mathscr{L}_0) \times \Gamma\left(\mathrm{Hom}(E_0, \$ \otimes \mathscr{L}_0)\right) \times (0, \pi/2)$ satisfies (2.10). Set*

$$\mathfrak{e} := \frac{r \int_{B_r} |\nabla_{A \otimes B} \Psi|^2}{\int_{\partial B_r} |\Psi|^2} + r^2 \|R_g\|_{L^\infty(B_r)} + r^2 \|F_B\|_{L^\infty(B_r)}$$

$$\text{and} \quad \tau := \frac{\tan(\alpha)}{\sqrt{\int_{\partial B_r} |\Psi|^2}}.$$

Let $\delta \in (0,1)$ and $\varepsilon > 0$. If

$$r^{1/2} \|F_A\|_{L^2(B_r)} \leqslant 1, \quad \mathfrak{e} \ll_{\delta, \varepsilon} 1 \quad \text{and} \quad \tau \ll_{\delta, \varepsilon} 1,$$

then

$$r^{1/2} \|F_A\|_{L^2(B_{(1-\delta)r})} \leqslant \varepsilon.$$

The statement of Proposition 5.3 is invariant under rescaling B_r, multiplying Ψ by a constant and changing α—hence, $\tan(\alpha)$—accordingly so that (2.10) still holds. Therefore, it suffices to consider the case $r = 1$ and $\int_{\partial B_r} |\Psi|^2 = 1$. Throughout the rest of this subsection assume the hypotheses of Proposition 5.3 with this normalisation.

Proposition 5.4. *There are constants* $0 < \lambda \leqslant \Lambda = \Lambda(\delta)$ *such that in* $B_{1-\delta}$

$$|\Psi| \leqslant \Lambda \quad \textit{and} \quad \textit{if } \mathfrak{e} \ll_\delta 1, \quad \textit{then} \quad |\Psi| \geqslant \lambda.$$

Proof. We proceed in three steps.

Step 1. *If* $\mathfrak{e} \leqslant 1$*, then for each* $x \in B_1$

$$|\Psi|^2(x) \lesssim d(x, \partial B_1)^{-2}.$$

In particular, $|\Psi| \leqslant \Lambda(\delta) = O(1/\delta)$.

We use a slight modification of the argument used to prove Proposition 2.1. It follows from (4.12) that $\|\Psi\|_{L^2(B_1)} = O(1)$ and thus $\||\Psi|\|_{W^{1,2}(B_1)} = O(1)$; hence, by Kato's inequality and Sobolev embedding we have $\|\Psi\|_{L^6(B_1)} = O(1)$.

Let G denote the Green's function for Δ on B_1. Fix $x \in B_1$ and set $f := G(x, \cdot)$. Then

$$f \lesssim \frac{1}{d(x,\cdot)} \quad \text{and} \quad |\nabla f| \lesssim \frac{1}{d(x,\cdot)^2}.$$

Apply Proposition 2.9 with f as above and $U = B_1 \backslash B_\sigma(x)$, and pass to the limit $\sigma = 0$ to obtain

$$|\Psi|^2(x) \lesssim \int_{B_1} \frac{|\Psi|^2}{d(x,\cdot)} + d(x, \partial B_1)^{-1} \int_{\partial B_1} \partial_r |\Psi|^2 + d(x, \partial B_1)^{-2}.$$

The first term is $O(1)$ since $\|1/d(x,\cdot)\|_{L^{3/2}(B_1)} = O(1)$. Applying Proposition 2.9 again with $f = 1$ and $U = B_1$ gives

$$\int_{\partial B_1} \partial_r |\Psi|^2 \lesssim \int_{B_1} |\Psi|^2 + |\nabla_A \Psi|^2 + \tau^{-2} |\mu(\Psi)|^2 = O(1).$$

Here we have also used that

$$\|\mu(\Psi)\|_{L^2(B_1)} = \tau^2 \|F_A\|_{L^2(B_1)} \leqslant \tau^2. \tag{5.5}$$

Step 2. *We have* $[|\Psi|]_{C^{0,1/4}(B_{1-\delta})} \lesssim_\delta \mathfrak{e}^{1/8}$.

Combining the Gagliardo–Nirenberg interpolation inequality

$$\|f\|_{L^4(B_{1-\delta})} \lesssim_\delta \|\nabla f\|^{3/4}_{L^2(B_{1-\delta})} \|f\|^{1/4}_{L^2(B_{1-\delta})} + \|f\|_{L^2(B_{1-\delta})},$$

with Kato's inequality, we obtain

$$
\begin{aligned}
\|\nabla_{A\otimes B}\Psi\|_{L^4(B_{1-\delta})} &\lesssim_\delta \|\nabla|\nabla_{A\otimes B}\Psi|\|_{L^2(B_{1-\delta})}^{3/4}\|\nabla_{A\otimes B}\Psi\|_{L^2(B_{1-\delta})}^{1/4} \\
&\quad + \|\nabla_{A\otimes B}\Psi\|_{L^2(B_{1-\delta})} \\
&\lesssim_\delta \mathfrak{e}^{1/8}.
\end{aligned} \tag{5.6}
$$

The asserted estimate now follows from Morrey's inequality combined with Kato's inequality.

Step 3. *There is a constant $\lambda > 0$ such that if $\mathfrak{e} \ll_\delta 1$, then in $B_{1-\delta}$*

$$|\Psi| \geqslant \lambda.$$

We know from Proposition 4.18 that

$$\int_{\partial B_{1-\delta}} |\Psi|^2 \gtrsim \int_{\partial B_1} |\Psi|^2 = 1,$$

which proves the lower bound on $|\Psi|$ when combined with Step 2. □

Proposition 5.7. *If $\mathfrak{e} \leqslant 1$, then*

$$\|\mu(\Psi)\|_{L^\infty(B_{1-\delta})} \lesssim_\delta \tau^{1/8}.$$

Proof. Using Kato's inequality, Proposition 5.4 and (5.6) we obtain

$$
\begin{aligned}
\|\nabla^2|\mu(\Psi)|\|_{L^2(B_{1-\delta})} &\lesssim \|\nabla^2_{A\otimes B}\Psi\|_{L^2(B_{1-\delta})}\|\Psi\|_{L^\infty(B_{1-\delta})} + \|\nabla_{A\otimes B}\Psi\|^2_{L^4(B_{1-\delta})} \\
&\lesssim_\delta 1.
\end{aligned}
$$

Hence, using the Gagliardo–Nirenberg interpolation inequality

$$\|\nabla f\|_{L^4(B_{1-\delta})} \lesssim \|\nabla^2 f\|_{L^2(B_{1-\delta})}^{7/8}\|f\|_{L^2(B_{1-\delta})}^{1/8} + \|f\|_{L^2(B_{1-\delta})}$$

and Morrey's inequality we obtain

$$\||\mu(\Psi)|\|_{C^{1/4}(B_{1-\delta})} \lesssim_\delta \||\mu(\Psi)|\|_{W^{1,4}(B_{1-\delta})} \lesssim_\delta \tau^{1/8}.$$ □

Proof of Proposition 5.3. By a straight-forward calculation

$$\mu(\mu(\Psi)\Psi, \Psi) = \frac{1}{2}|\Psi|^2\mu(\Psi) + \mu(\Psi)\circ\mu(\Psi) - \frac{1}{2}\operatorname{tr}(\mu(\Psi)\circ\mu(\Psi))\,\mathrm{id}_{\$}.$$

Using this and the Weitzenböck formula (2.6) we get

$$
\begin{aligned}
\nabla^*\nabla\mu(\Psi) &= 2\mu(\nabla^*_{A\otimes B}\nabla_{A\otimes B}\Psi, \Psi) - 2\left\langle\mu(\nabla_{A\otimes B}\Psi, \nabla_{A\otimes B}\Psi)\right\rangle \\
&= -\left(\tau^{-2}|\Psi|^2 + \frac{s}{2}\right)\mu(\Psi) \\
&\quad + 2\tau^{-2}\mu(\Psi)\circ\mu(\Psi) - \tau^{-2}\operatorname{tr}(\mu(\Psi)\circ\mu(\Psi))\,\mathrm{id}_{\$} \\
&\quad - 2\mu(F_B\Psi, \Psi) - 2\left\langle\mu(\nabla_{A\otimes B}\Psi, \nabla_{A\otimes B}\Psi)\right\rangle.
\end{aligned}
$$

where $\langle\cdot,\cdot\rangle$ denotes the contraction $T^*M \otimes T^*M \to \mathbf{R}$.

Fix a cut-off function χ which is supported in $B_{1-\delta/2}$ and is equal to one in $B_{1-\delta}$. Then the above yields

$$\begin{aligned}\int \chi|\nabla\mu(\Psi)|^2 + \left(\tau^{-2}|\Psi|^2 + \frac{s}{2}\right)\chi|\mu(\Psi)|^2 \\ = \int 2\chi\tau^{-2}\langle\mu(\Psi)\circ\mu(\Psi),\mu(\Psi)\rangle - 2\chi\langle\mu(F_B\Psi,\Psi),\mu(\Psi)\rangle \\ - 2\chi\langle\langle\mu(\nabla_{A\otimes B}\Psi,\nabla_{A\otimes B}\Psi)\rangle,\mu(\Psi)\rangle \\ - \langle\nabla^{A\otimes B}_{\nabla\chi}\mu(\Psi),\mu(\Psi)\rangle\end{aligned}$$

Since $\|\mu(\Psi)\|_{L^\infty(B_{1-\delta/2})} \lesssim_\delta \tau^{1/8}$, the first term on the right hand side can be bounded by

$$c_\delta\tau^{-2+1/8}\int \chi|\mu(\Psi)|^2.$$

Thus, using Proposition 5.4 and (5.6), for $\mathfrak{e} \ll_\delta 1$ and $\tau \ll_\delta 1$, we obtain

$$\begin{aligned}\int \chi|\mu(\Psi)|^2 &\lesssim_\delta \tau^2\int \left(|F_B||\Psi|^2 + |\nabla_{A\otimes B}\Psi|^2 + |\Psi||\nabla_{A\otimes B}\Psi|\right)|\mu(\Psi)| \\ &\lesssim_\delta \tau^4(\mathfrak{e} + \mathfrak{e}^{1/4}).\end{aligned}$$

This implies the assertion because $F_A = \tau^{-2}\mu(\Psi)$. □

5.2 Proof of Proposition 5.1

If the assertion does not hold, then there exist solutions $(A,\Psi,\alpha) \in \mathscr{A}(\mathscr{L}) \times \Gamma\left(\mathrm{Hom}(E, \$ \otimes \mathscr{L})\right) \times (0,\pi/2]$ of (1.4) and $x \in M$ with $\rho \leqslant \varepsilon$ and $\mathrm{N}(50\rho) \leqslant \varepsilon$ for arbitrarily small $\varepsilon > 0$. The next four steps show that this is impossible.

Step 1. *There is a point $x' \in B_{2\rho(x)}(x)$ such that*

$$\rho(x') \leqslant \rho(x) \quad \textit{and} \quad \rho(x') \leqslant 2\min\{\rho(y) : y \in B_{\rho(x')}(x')\}.$$

Construct a sequence x_k inductively. Set $x_0 := x$ and assume that x_k has been constructed. If

$$\rho(x_k) \leqslant 2\min\{\rho(y) : y \in B_{\rho(x_k)}(x_k)\},$$

then we set $x' := x_k$. Otherwise we choose $x_{k+1} \in B_{\rho(x_k)}(x_k)$ such that

$$\rho(x_{k+1}) < \frac{1}{2}\rho(x_k).$$

By construction we have $\rho(x_{k+1}) < \frac{1}{2^k}\rho(x)$. Since $\rho(\cdot)$ is bounded below for a fixed (A,Ψ,α), this sequence must terminate for some k. Note that

$$d(x,x') \leqslant \sum_{i=0}^{k}\rho(x_i) \leqslant 2\rho(x).$$

Step 2. *For each $y \in B_{\rho(x')}(x')$ we have $\rho(y) \lesssim \varepsilon$ and $\mathrm{N}_y(\rho(y)) \lesssim \varepsilon$.*

If $y \in B_{\rho(x')}(x')$, then $B_{2\rho(x')}(y) \supset B_{\rho(x')}(x')$; hence,

$$\int_{B_{2\rho(x')}(y)} |F_A|^2 \geqslant \frac{1}{\rho(x')} > \frac{1}{2\rho(x')}$$

and therefore $\rho(y) < 2\rho(x') \leqslant 2\rho(x) \lesssim \varepsilon$. Since $y \in B_{5\rho(x)}(x)$, we can apply Proposition 4.21 with $r = 5\rho(x)$ to deduce that $\mathrm{N}_y(\rho(y)) \leqslant e^{O(\varepsilon^2)}\mathrm{N}_y(25\rho(x)) + O(\varepsilon^2) \lesssim \mathrm{N}_x(50\rho(x)) + O(\varepsilon^2) \lesssim \varepsilon$.

Step 3. *There exists a finite set $\{y_1, \dots, y_k\} \subset B_{\rho(x')}(x')$ with $k = O(1)$ such that*

$$\bigcup B_{\rho(y_i)/2}(y_i) \supset B_{\rho(x')}(x').$$

It follows from the first step that for each $y \in B_{\rho(x')}(x')$ we have $\rho(y) \geqslant \frac{1}{2}\rho(x')$. This implies the existence of a finite set $\{y_i\}$ with the desired properties.

Step 4. *We prove the proposition.*

By Proposition 5.2 and the previous steps

$$\int_{B_{\rho(y_i)/2}(y_i)} |F_A|^2 \lesssim \frac{\varepsilon}{\rho(y_i)} \lesssim \frac{\varepsilon}{\rho(x')};$$

hence,

$$\int_{B_{\rho(x')}(x')} |F_A|^2 \lesssim \frac{\varepsilon}{\rho(x')}.$$

If $\varepsilon \ll 1$, this contradicts the definition of $\rho(x')$. □

6 Convergence on $M\backslash Z$

In this section we prove the following convergence result, which completes the proof of Theorem 1.5 (except for the statement regarding the size of Z).

Proposition 6.1. *In the situation of Theorem 1.5 if $\limsup \alpha_i = 0$ and with $|\Psi|$ as in Proposition 2.3 after passing to a further subsequence the following hold:*

1. *There is a constant $\gamma > 0$ such that $|\Psi_i|$ converges to $|\Psi|$ in $C^{0,\gamma}$. In particular, the set $Z := |\Psi|^{-1}(0)$ is closed.*
2. *There is a flat connection A on $\mathscr{L}|_{M\backslash Z}$ with monodromy in $\mathbf{Z}_2$ and $\Psi \in \Gamma\left(M\backslash Z, \mathrm{Hom}(E, \$ \otimes \mathscr{L})\right)$ such that $(A, \Psi, 0)$ solves (1.4). On $M\backslash Z$ up to gauge transformations A_i converges weakly in $W^{1,2}_{\mathrm{loc}}$ to A and Ψ_i converges weakly in $W^{2,2}_{\mathrm{loc}}$ to Ψ.*

To prove this we need the following result.

Proposition 6.2. *There is a constant $\gamma > 0$ such that whenever $(A, \Psi, \alpha) \in \mathscr{A}(\mathscr{L}) \times \Gamma\left(\mathrm{Hom}(E, \$ \otimes \mathscr{L})\right) \times (0, \pi/2]$ is a solution of* (1.4)*, then* $[|\Psi|]_{C^{0,\gamma}} = O(1)$.

Proof. Let $x \neq y \in M$. We need to uniformly control

$$\frac{||\Psi|(x) - |\Psi|(y)|}{d(x,y)^\gamma}$$

for some $\gamma > 0$. Take $\omega > 0$ as in Proposition 4.1. Without loss of generality we can assume that $d(x,y) \leqslant \omega$ and $0 \neq v := |\Psi|(x) \geqslant |\Psi|(y)$. It follows from Proposition 4.1 that

$$\rho(x) \gtrsim \min\{1, v^{1/\omega}\}. \tag{6.3}$$

We distinguish two cases.

Case 1. $d(x,y)^{1/2} \leqslant \rho(x)/2$.

By combining Proposition 3.3 with Sobolev embedding, Morrey's inequality with Kato's inequality we obtain

$$\frac{||\Psi|(x) - |\Psi|(y)|}{d(x,y)^{1/2}} \lesssim \|\nabla_{A\otimes B}\Psi\|_{L^6(B_{\rho(x)/2})} \lesssim \rho(x)^{-1/2} \lesssim d(x,y)^{-1/4};$$

hence,

$$\frac{||\Psi|(x) - |\Psi|(y)|}{d(x,y)^{1/4}} = O(1).$$

Case 2. $d(x,y)^{1/2} > \rho(x)/2$.

If $v \geqslant 1$, then by (6.3) we are in Case 1. Thus $v < 1$ and it follows from (6.3) that

$$|\Psi|(y) \leqslant |\Psi|(x) \lesssim \rho(x)^\omega \lesssim d(x,y)^{\omega/2};$$

hence,

$$\frac{||\Psi|(y) - |\Psi|(x)|}{d(x,y)^{\omega/2}} = O(1).$$

This proves the proposition with $\gamma := \min\{\frac{1}{4}, \frac{\omega}{2}\}$. □

Proof of Proposition 6.1. Proposition 6.2 immediately implies the first part of the proposition. We prove the second part. If $x \in M \backslash Z$, then, by Proposition 4.1, after passing to a subsequence the critical radius $\rho_i(x)$ of (A_i, Ψ_i, α_i) is bounded below by a constant, say, $2R > 0$ depending only on $|\Psi|(x)$. By Proposition 6.2 we can also assume that $|\Psi_i|$ is bounded away from zero on $B_{2R}(x)$, after possibly making R smaller. Combining Proposition 5.3 and Proposition 4.19 yields L^2–bounds on F_{A_i} on balls covering $B_R(x)$; hence, by Proposition 3.3, $W^{2,2}_{A_i}$–bounds on Ψ_i. After putting A_i in

Uhlenbeck gauge on $B_R(x)$ and passing to a subsequence the sequence (A_i, Ψ_i) converges weakly in $W^{1,2} \oplus W^{2,2}$ to a limit (A, Ψ). The pair (A, Ψ) satisfies

$$\not{D}_{A\otimes B}\Psi = 0 \quad \text{and} \quad \mu(\Psi) = 0.$$

The local gauge transformations can be patched to obtain a global gauge transformation on $M\backslash Z$, see [DK90, Section 4.2.2].

The fact that A has monodromy in $\mathbf{Z}_2$ follows from the discussion in Appendix A. □

7 Z is nowhere-dense

Since $\int_M |\Psi|^2 = 1$, we know that Z cannot be the entire space. To obtain more precise information on Z it turns out to be helpful to apply the ideas from Section 4 to the limit (A, Ψ). Fix $x \in M$ and define functions $H, h\colon [0, r_0] \to [0, \infty)$ by

$$H(r) \coloneqq \int_{B_r(x)} |\nabla_{A\otimes B}\Psi|^2 \quad \text{and}$$
$$h(r) \coloneqq \int_{\partial B_r(x)} |\Psi|^2.$$

Here we extend $|\nabla_{A\otimes B}\Psi|$ by defining it to be zero on Z. If $h(r) > 0$, define

$$\textsc{n}(r) \coloneqq \frac{rH(r)}{h(r)}.$$

Proposition 7.1. *Denote by h_i, H_i the (A_i, Ψ_i, α_i) version of h and H defined in Definition 4.2. The sequences of functions h_i and H_i converge uniformly to h and H, respectively. In particular, $\textsc{n}_i(r) \to \textsc{n}(r)$ whenever $h(r) > 0$.*

Let us first explain how this implies the following.

Proposition 7.2. *Z is nowhere-dense.*

Proof. Choose $R \geqslant 0$ as large as possible, but so that $B_R(x) \subset Z$. We know that R is finite, because Z is compact. By replacing x with a point close to the boundary of $B_R(x)$ we can assume that $R \ll 1$. By construction of R there is an $\varepsilon \ll 1$ such that $h(R + \varepsilon) > 0$. In particular, $\textsc{n}(R + \varepsilon)$ is defined. It follows from Proposition 4.18 and Proposition 7.1 that $R = 0$. □

Proof of Proposition 7.1. That h_i converges uniformly to h is a direct consequence of the $C^{0,\gamma}$ convergence of $|\Psi_i|$. The proof of the corresponding statement for H_i has three steps.

Step 1. *For $\varepsilon \in (0, 1/2]$ set $Z_\varepsilon \coloneqq |\Psi|^{-1}([0, \varepsilon])$. The sequence of functions*

$$H_{\varepsilon,i}(r) \coloneqq \int_{B_r(x)\backslash Z_\varepsilon} |\nabla_{A_i\otimes B}\Psi_i|^2 + \tan(\alpha_i)^{-2}|\mu(\Psi_i)|^2$$

converges uniformly to

$$H_\varepsilon(r) := \int_{B_r\backslash Z_\varepsilon} |\nabla_{A\otimes B}\Psi|^2.$$

This follows from the facts that $\tan(\alpha_i)^{-1}\mu(\Psi_i) = \tan(\alpha_i)F_{A_i}$ converges to zero in $L^2(M\backslash Z_\varepsilon)$ and $\nabla_{A_i\otimes B}\Psi_i$ converges to $\nabla_{A\otimes B}\Psi$ in $L^2(M\backslash Z_\varepsilon)$, see Proposition 6.1.

Step 2. *There exists a* $\lambda > 0$ *such that*

$$\int_{Z_\varepsilon} |\nabla_{A_i\otimes B}\Psi_i|^2 + \tan(\alpha_i)^{-2}|\mu(\Psi_i)|^2 = O(\varepsilon^\lambda).$$

Fix a cut-off function $\chi\colon \mathbf{R} \to [0,1]$ with $\chi(t) = 1$ for $t \leqslant 1$ and $\chi(t) = 0$ for $t \geqslant 2$. Applying Proposition 2.9 with $f = \chi(\varepsilon^{-1}|\Psi_i|)$ and $U = M$, integrating the resulting term with $\Delta|\Psi|$ by parts once and using Kato's inequality yields

$$\int_{Z_\varepsilon} |\nabla_{A_i\otimes B}\Psi_i|^2 + \tan(\alpha_i)^{-2}|\mu(\Psi_i)|^2 \leqslant c\varepsilon^2 + c\int_{Z_{2\varepsilon}\backslash Z_\varepsilon} |\nabla_{A_i\otimes B}\Psi_i|^2.$$

Denoting

$$f(\varepsilon) := \int_{Z_\varepsilon} |\nabla_{A_i\otimes B}\Psi_i|^2 + \tan(\alpha_i)^{-2}|\mu(\Psi_i)|^2$$

this can be written as

$$f(\varepsilon) \leqslant \sigma(\varepsilon^2 + f(2\varepsilon))$$

with $\sigma := c/(1+c)$. Since f is bounded above and we can assume that $\sigma \geqslant 1/2$,

$$\begin{aligned} f(\varepsilon) &\leqslant \sigma\varepsilon^2\sum_{i=0}^{k-1}(4\sigma)^i + \sigma^k f(2^k\varepsilon) \\ &\leqslant \varepsilon^2\sigma\left(\frac{(4\sigma)^{k-1}-1}{4\sigma-1}\right) + c\sigma^k \\ &\lesssim \varepsilon^2(4\sigma)^k + \sigma^k. \end{aligned}$$

With $k := \lfloor -\log\varepsilon/\log 2\rfloor$ this gives

$$f(\varepsilon) \lesssim \varepsilon^{2-\log(4\sigma)/\log 2} + \varepsilon^{-\log\sigma/\log 2} \lesssim \varepsilon^\lambda$$

for some $\lambda > 0$ depending on σ only, since $\log(4\sigma)/\log 2 < 2$.

Step 3. *The sequence of functions* H_i *converges uniformly to* H.

Both $|H_\varepsilon(r) - H(r)|$ and $|H_{\varepsilon,i}(r) - H_i(r)|$ converge uniformly to zero as ε goes to zero, the former by monotone convergence and the latter by Step 2; hence, the desired convergence follows immediately from Step 1. □

A Fueter sections of bundles of moduli spaces of ASD instantons

Recall from [DK90, Section 3.3] that if E denotes a Hermitian vector space of dimension n with fixed determinant, $\slashed{S}^+$ denotes the positive spin representation of $\mathrm{Spin}(4)$ and $\mathscr{L}$ is a Hermitian vector space of dimension one, then

$$\big(\mathrm{Hom}(E, \slashed{S}^+ \otimes \mathscr{L})\backslash\{0\}\big)/\!\!/\!\!/\mathrm{U}(1) = \mathring{M}_{1,n}$$

the moduli space of centred framed charge one $\mathrm{SU}(n)$ ASD instantons on $\mathbf{R}^4$.

In the situation of Section 1 we have bundles of the above data (which we denote by the same letters) and can construct the bundle

$$\mathfrak{M} := (\mathfrak{s} \times \mathrm{SU}(E)) \times_{\mathrm{Spin}(3)\times\mathrm{SU}(n)} \mathring{M}_{1,n}.$$

Here $\mathrm{SU}(E)$ is the principal $\mathrm{SU}(n)$–bundle of oriented orthonormal frames of E and $\mathrm{Spin}(3)$ acts via the inclusion of the first factor in $\mathrm{Spin}(4) = \mathrm{Spin}_+(3) \times \mathrm{Spin}_-(3)$. Using the connections on $\mathfrak{s}$ and E we can associate to every section $\mathfrak{I} \in \Gamma(\mathfrak{M})$ its covariant derivative $\nabla\mathfrak{I} \in \Omega^1(\mathfrak{I}^*V\mathfrak{M})$. Here $V\mathfrak{M} := (\mathfrak{s} \times \mathrm{SU}(E)) \times_{\mathrm{Spin}(3)\times\mathrm{SU}(n)} T\mathring{M}_{1,n}$ is the vertical tangent bundle of $\mathfrak{M}$. Moreover, there is a Clifford multiplication $\gamma\colon TM \otimes \mathfrak{I}^*V\mathfrak{M} \to \mathfrak{I}^*V\mathfrak{M}$. Therefore, there is a natural non-linear Dirac operator $\mathfrak{F}$, called the *Fueter operator*, which assigns to a section $\mathfrak{I} \in \Gamma(\mathfrak{M})$ the vertical vector field

$$\mathfrak{F}\mathfrak{I} := \sum_{i=1}^{3} \gamma(e_i)\nabla_{e_i}\mathfrak{I} \in \Gamma(\mathfrak{I}^*V\mathfrak{M}).$$

Proposition A.1. *If $A \in \mathscr{A}(\mathscr{L})$ and $\Psi \in \Gamma(M, \mathrm{Hom}(E, \slashed{S} \otimes \mathscr{L}))$ is a solution of*

$$\begin{aligned} \slashed{D}_{A\otimes B}\Psi &= 0 \quad \textit{and} \\ \mu(\Psi) &= 0. \end{aligned} \tag{A.2}$$

and Ψ vanishes nowhere, then the induced section $\mathfrak{I} \in \Gamma(\mathfrak{M})$ solves $\mathfrak{F}\mathfrak{I} = 0$. Conversely, each Fueter section $\mathfrak{I} \in \Gamma(\mathfrak{M})$ lifts to a solution (A, Ψ) of (A.2) for some $\mathscr{L}$.

The proof is essentially the same as that of [Hay12, Proposition 4.1]. It is worthwhile to explain how $\mathscr{L}$ and A are recovered from $\mathfrak{I}$: the $\mathrm{U}(1)$–bundle $\mu^{-1}(0) \to \mathring{M}_{1,n}$ has a canonical connection given by orthogonal projection along the $\mathrm{U}(1)$–orbits; hence, the $\mathrm{U}(1)$–bundle $\underline{\mathscr{L}} := (\mathfrak{s} \times \mathrm{SU}(E)) \times_{\mathrm{Spin}(3)\times\mathrm{SU}(n)} \mu^{-1}(0) \to \mathfrak{M}$ inherits a connection $\underline{A}$; and, finally, $\mathscr{L}$ and A are obtained via pullback:

$$\mathscr{L} = \mathfrak{I}^*\underline{\mathscr{L}} \quad \text{and} \quad A = \mathfrak{I}^*\underline{A}.$$

The following gives more information about A.

Proposition A.3. *Let $A \in \mathscr{A}(\mathscr{L})$ and $\Psi \in \Gamma(M, \mathrm{Hom}(E, \slashed{S} \otimes \mathscr{L}))$ be a solution of (A.2). Denote $Z := \Psi^{-1}(0)$. In this situation the following hold true:*

1. *$F := \operatorname{coim}(\Psi) = \ker(\Psi)^\perp$ is a rank 2 subbundle of $E|_{M\backslash Z}$.*

2. *The bundle $\mathscr{K} := \det F$ has a square root $\sqrt{\mathscr{K}}$. In particular, $\mathring{F} := F \otimes \mathscr{K}^{-1/2}$ is an $\mathrm{SU}(2)$–bundle.*

3. *The connection induced on $\mathscr{L}|_{M\backslash Z} \otimes \mathscr{K}^{1/2}$ and the induced section $\Phi \in \Gamma(M\backslash Z, \operatorname{Hom}(\mathring{F}, \$ \otimes \mathscr{L} \otimes \mathscr{K}^{1/2}))$ satisfy (A.2) over $M\backslash Z$ with respect to the induced connection on $\mathring{F}$. We have $|\Phi| = |\Psi|$ over $M\backslash Z$ and, hence, $|\Phi|$ extends as a continuous function over M and $Z = |\Phi|^{-1}(0)$.*

4. *The induced connection on $\mathscr{L} \otimes \mathscr{K}^{1/2}$ is flat and has $\mathbf{Z}_2$–monodromy.*

Proof. For each $x \in M$, $\mu^{-1}(0)\backslash\{0\} \subset \operatorname{Hom}(E, \$ \otimes \mathscr{L}))_x$ is acted upon transitively by $\mathbf{R}_+ \times \mathrm{U}(E_x)$. In particular, it can be checked directly that for one (hence for all) non-zero $\Psi \in \mu^{-1}(0)$ we have $\operatorname{rk}\Psi = 2$.

The induced section $\Phi \in \Gamma(M, \operatorname{Hom}(F, \$ \otimes \mathscr{L}))$ defines an isomorphism $F \cong \$ \otimes \mathscr{L}$, hence $\det F \cong \det(\$ \otimes \mathscr{L}) \cong \mathscr{L}^{\otimes 2}$. This implies (2).

The assertion made in (3) is a consequence of (A, Ψ) satisfying (A.2). Thus we are left with proving (4) in the case $n = 2$. In this case, $F = E$, $\mathscr{K}$ is trivial and $\Phi = \Psi$. To see that A is flat with monodromy in $\mathbf{Z}_2$ note that the same is true for the canonical connection on $\mu^{-1}(0) \to \mathring{M}_{1,2}$: note that $\mathbf{R}_+ \times \mathrm{U}(2)$ acts transitively on $\mu^{-1}(0)$, and the horizontal distribution is preserved by $\mathbf{R}_+ \times \mathrm{SU}(2)$ and therefore integrable, i.e., the canonical connection is flat. Since $\pi_1(\mathring{M}_{1,2}) = \mathbf{Z}_2$, the monodromy of the canonical connection lies in $\mathbf{Z}_2$. □

Remark A.4. If $\mathscr{L} \otimes \mathscr{K}^{1/2}$ carries a flat connection with monodromy in $\mathbf{Z}_2$, then it must be the complexification of a real line bundle $\mathfrak{l}$. Solutions to (1.4) with Spin–structure $\mathfrak{s}$ and $\mathrm{U}(1)$–bundle $\mathscr{L}$ are in one-to-one correspondence with solutions with Spin–structure $\mathfrak{s} \otimes \mathfrak{l}$ and $\mathrm{U}(1)$–bundle $\mathscr{L} \otimes (\mathfrak{l} \otimes \mathbf{C})$. Therefore we can assign to each Fueter section $\mathfrak{I}$ the unique Spin–structure $\mathfrak{s}$ which makes $\mathscr{L} \otimes \mathscr{K}^{1/2}$ trivial.

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