Hecke modifications of Higgs bundles and the extended Bogomolny equation

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Abstract

We establish a Kobayashi–Hitchin correspondence between solutions of the extended Bogomolny equation with a Dirac type singularity and Hecke modifications of Higgs bundles. This correspondence was conjectured by Witten [Wit18, p. 668] and plays an important role in the physical description of the the geometric Langlands program in terms of S–duality for $\mathcal{N} = 4$ super Yang–Mills theory in four dimensions.

1 Introduction

Kapustin and Witten [KW07] describe the geometric Langlands program in terms of S–duality for $\mathcal{N} = 4$ super Yang–Mills theory in four dimensions. At the heart of their description lies the observation that every solution of the Bogomolny equation with a Dirac type singularity on $[0, 1] \times \Sigma$ gives rise to a Hecke modification of a holomorphic bundle over the Riemann surface Σ via a scattering map construction [KW07, Section 9; Hur85]. Moreover, they anticipated that this construction establishes a bijection between a suitable moduli space of singular monopoles and the moduli space of Hecke modifications—similar to the Kobayashi–Hitchin correspondence [Don85; Don87; UY86; LT95]. Their conjecture has been proved by Norbury [Nor11]; see also Charbonneau and Hurtubise [CH11] and Mochizuki [Moc17].

In a recent article, Witten [Wit18] elaborates on the physical description of the geometric Langlands program and emphasizes the importance of the relation between solutions to the *extended* Bogomolny equation with a Dirac type singularity on $[0, 1]\times\Sigma$ and Hecke modifications of *Higgs* bundles. While Hecke modifications of holomorphic bundles have been studied intensely for quite some time (see, e.g., [PS86; Zhu17]), interest in Hecke modifications of Higgs bundles has only emerged recently. They do appear, for example, in Nakajima's recent work on a mathematical definition of Coulomb branches of 3–dimensional $\mathcal{N} = 4$ gauge theories [Nak17, Section 3].

The purpose of this article is to (a) give a precise statement of the Kobayashi–Hitchin correspondence conjectured by Witten and (b) establish this correspondence. The upcoming four sections review the notion of a Hecke modification of a Higgs bundle, the extended Bogomolny equation, Dirac type singularities, and the scattering map construction. The main result of this article is stated as Theorem 5.10. The remaining five sections contain the proof of this result.

Our proof, like Norbury's, heavily relies on the work of Simpson [Sim88]. However, unlike Norbury, we cannot make use of the extensive prior work on Dirac type singularities for solutions of the Bogomolny equation [Kro85; Pau98; MY17]. Instead, our singularity analysis is based on ideas from recent work on tangent cones of singular Hermitian Yang–Mills connections [JSW18; CS17]. Theorem 5.10 can be easily generalized to a Kobayashi–Hitchin correspondence between solutions of the extended Bogomolny equation with multiple Dirac type singularities and sequences of Hecke modifications of Higgs bundles. This result is stated as Theorem A.3 and proved in Appendix A. Moreover, although we do not provide details here, both of these results can be further generalized to $G^{\mathbf{C}}$ Higgs bundles by fixing an embedding $G \subset \mathrm{U}(r)$, see [Sim88, Proof of Proposition 8.2].

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2 Hecke modifications of Higgs bundles

In this section, we briefly recall the notion of a Hecke modification of a Higgs bundle. We refer the reader to [Wit18, Section 4.5] for a more extensive discussion. Throughout this section, let (Σ, I) be a closed Riemann surface and denote its canonical bundle by K_Σ.

Definition 2.1. A **Higgs bundle** over Σ is a pair $(\mathscr{E}, \varphi)$ consisting of a holomorphic vector bundle $\mathscr{E}$ over Σ and a holomorphic 1–form $\varphi \in H^0(\Sigma, K_\Sigma \otimes \mathrm{End}(\mathscr{E}))$ with values in $\mathrm{End}(\mathscr{E})$. •

Let (E, H) be a Hermitian vector bundle over Σ. Given a holomorphic structure $\bar\partial$ on E, there exists a unique unitary connection $A \in \mathscr{A}(E, H)$ satisfying

$$\nabla_A^{0,1} = \bar\partial;$$

see, e.g., [Che95, Section 6]. Furthermore, every $\varphi \in \Omega^{1,0}(\Sigma, \mathrm{End}(E))$ can uniquely be written as

$$\varphi = \frac{1}{2}(\phi - iI\phi)$$

with $\phi \in \Omega^1(\Sigma, \mathfrak{u}(E, H))$. Here I is the complex structure on Σ and $\mathfrak{u}(E, H)$ denotes the bundle of skew-Hermitian endomorphism of (E, H). It follows from the Kähler identities that φ is holomorphic if and only if

$$\mathrm{d}_A\phi = 0 \quad \text{and} \quad \mathrm{d}_A^*\phi = 0. \tag{2.2}$$

Remark 2.3. Hitchin [Hit87, Theorem 2.1 and Theorem 4.3] proved that a Higgs bundle $(\mathscr{E}, \varphi)$ of rank $r := \mathrm{rk}\,\mathscr{E}$ admits a Hermitian metric H such that (A, ϕ) satisfies **Hitchin's equation**

$$F_A^\circ - \frac{1}{2}[\phi \wedge \phi] = 0, \quad \mathrm{d}_A\phi = 0, \quad \text{and} \quad \mathrm{d}_A^*\phi = 0 \tag{2.4}$$

if and only if it is μ–polystable. Here $F_A^\circ \coloneqq F_A - \frac{1}{r}\operatorname{tr}(F_A)\mathrm{id}_E$. Furthermore, if $(\mathscr{E}, \varphi)$ is μ–stable, then imposing the additional condition that H induces a given Hermitian metric on $\det \mathscr{E}$ makes it unique. ♣

Definition 2.5. Let $(\mathscr{E}, \varphi)$ be a Higgs bundle over Σ of rank r. Let $z_0 \in \Sigma$ and $\mathbf{k} = (k_1, \dots, k_r) \in \mathbf{Z}^r$ satisfying

$$k_1 \leqslant k_2 \leqslant \cdots \leqslant k_r. \tag{2.6}$$

A **Hecke modification** of $(\mathscr{E}, \varphi)$ at z_0 of type $\mathbf{k}$ is a Higgs bundle $(\mathscr{F}, \chi)$ over Σ together with an isomorphism

$$\eta\colon\ (\mathscr{E}, \varphi)|_{\Sigma\setminus\{z_0\}} \cong (\mathscr{F}, \chi)|_{\Sigma\setminus\{z_0\}}$$

of Higgs bundles which, in suitable holomorphic trivializations near z_0, is given by

$$\operatorname{diag}(z^{k_1}, \dots, z^{k_r}).$$

An **isomorphism** between two Hecke modifications $(\mathscr{F}_1, \chi_1; \eta_1)$ and $(\mathscr{F}_2, \chi_2; \eta_2)$ of $(\mathscr{E}, \varphi)$ is an isomorphism

$$\zeta\colon\ (\mathscr{F}_1, \chi_1) \to (\mathscr{F}_2, \chi_2)$$

such that

$$\eta_1 = \eta_2\zeta.$$

We denote by

$$\mathscr{M}^{\mathrm{Hecke}}(\mathscr{E}, \varphi; z_0, \mathbf{k})$$

the set of all isomorphism classes of Hecke modifications of $(\mathscr{E}, \varphi)$ at z_0 of type $\mathbf{k}$. •

Remark 2.7. If $\varphi = 0$, then the above reduces to the classical notion of a Hecke modification of a holomorphic vector bundle. ♣

3 Singular solutions of the extended Bogomolny equation

Throughout this section, let M be an oriented Riemannian 3–manifold (possibly with boundary) and let (E, H) be a Hermitian vector bundle over M.

Definition 3.1. The **extended Bogomolny equation** is the following partial differential equation for $A \in \mathscr{A}(E, H)$, $\phi \in \Omega^1(M, \mathfrak{u}(E, H))$, and $\xi \in \Omega^0(M, \mathfrak{u}(E, H))$:

$$\begin{aligned} F_A - \frac{1}{2}[\phi \wedge \phi] &= *\mathrm{d}_A\xi, \\ \mathrm{d}_A\phi - *[\xi, \phi] &= 0, \quad \text{and} \\ \mathrm{d}_A^*\phi &= 0. \end{aligned} \tag{3.2}$$

•

Remark 3.3. The extended Bogomolny equation arises from the Kapustin–Witten equation [KW07] by dimensional reduction. It can be thought of as a complexification of the Bogomolny equation. In fact, for $\phi = 0$, it reduces to the Bogomolny equation. ♣

In this article, we are exclusively concerned with singular solutions of (3.2). The following example is archetypical.

Example 3.4. Let $k \in \mathbf{Z}$. The holomorphic line bundle $\mathcal{O}_{\mathbf{C}P^1}(k) \to \mathbf{C}P^1 \cong S^2$ admits a metric H_k whose associated connection B_k satisfies

$$F_{B_k} = -\frac{ik}{2}\mathrm{vol}_{S^2}.$$

Denote by $\pi\colon \mathbf{R}^3\backslash\{0\} \to S^2$ the projection map and denote by $r\colon \mathbf{R}^3 \to [0,\infty)$ the distance to the origin.

Given $\mathbf{k} \in \mathbf{Z}^r$ satisfying (2.6), set

$$(E_{\mathbf{k}}, H_{\mathbf{k}}) \coloneqq \bigoplus_{i=1}^{r} \pi^*(\mathcal{O}_{\mathbf{C}P^1}(k_i), H_{k_i}), \quad A_{\mathbf{k}} \coloneqq \bigoplus_{i=1}^{r} \pi^* B_k, \quad \text{and} \quad \xi_{\mathbf{k}} \coloneqq \frac{1}{2r}\,\mathrm{diag}(ik_1, \ldots, ik_r).$$

The pair $(A_{\mathbf{k}}, \xi_{\mathbf{k}})$ is called the **Dirac monopole** of type $\mathbf{k}$. It satisfies the **Bogomolny equation**

$$F_{A_{\mathbf{k}}} = *\mathrm{d}_{A_{\mathbf{k}}}\xi_{\mathbf{k}}$$

and thus (3.2) with $\phi = 0$. ♠

Henceforth, we suppose that $\bar{M}$ is an oriented Riemannian 3–manifold, $p \in \bar{M}$ is an interior point, and M is the complement of p in $\bar{M}$. Define $r\colon M \to (0,\infty)$ by

$$r(x) \coloneqq d(x,p).$$

Furthermore, we fix $\mathbf{k} \in \mathbf{Z}^r$ satisfying (2.6).

Definition 3.5. A **framing** of (E,H) at p of type $\mathbf{k}$ is an isometry of Hermitian vector bundles

$$\Psi\colon\ \exp_p^*(E,H)|_{B_\rho(0)} \to (E_{\mathbf{k}}, H_{\mathbf{k}})|_{B_\rho(0)}$$

for some $\rho > 0$. •

Definition 3.6. Let Ψ be a framing of (E,H) at p of type $\mathbf{k}$. A solution (A,ϕ,ξ) of (3.2) on (E,H) is said to have a **Dirac type singularity** at p of type $\mathbf{k}$ if there exists an $\alpha > 0$ such that for every $k \in \mathbf{N}_0$

$$\nabla^k_{A_{\mathbf{k}}}(\Psi_*A - A_{\mathbf{k}}) = O(r^{-k-1+\alpha}), \quad \nabla^k_{A_{\mathbf{k}}}\Psi_*\phi = O(r^{-k}), \quad \text{and} \quad \nabla^k_{A_{\mathbf{k}}}(\Psi_*\xi - \xi_{\mathbf{k}}) = O(r^{-k-1+\alpha}).$$

A gauge transformation $u \in \mathcal{G}(E,H)$ is called **singularity preserving** if there exists a $u_p \in \mathcal{G}(E_{\mathbf{k}}, H_{\mathbf{k}})$ satisfying

$$\nabla_{A_{\mathbf{k}}} u_p = 0 \quad \text{and} \quad (u_p)_*\xi_{\mathbf{k}} = \xi_{\mathbf{k}}$$

and an $\alpha > 0$ such that for every $k \in \mathbf{N}_0$

$$\nabla^k_{A_{\mathbf{k}}}(\Psi_*u - u_p) = O(r^{-k+\alpha}).$$ •

4 The extended Bogomolny equation over $[0,1] \times \Sigma$

Throughout the remainder of this article, we assume that the following are given:

(1) a closed Riemann surface (Σ, I),

(2) a Hermitian vector bundle (E_0, H_0) over Σ,

(3) a solution (A_0, ϕ_0) of (2.2),

(4) $(y_0, z_0) \in (0,1) \times \Sigma$, and

(5) $\mathbf{k} \in \mathbf{Z}^r$ satisfying (2.6).

Set

$$M \coloneqq [0,1] \times \Sigma \backslash \{(y_0, z_0)\}$$

Proposition 4.1. *Given the above data, there exists a Hermitian vector bundle (E, H) over M whose restriction to $\{0\} \times \Sigma$ is isomorphic to (E_0, H_0) together with a framing Ψ at (y_0, z_0) of type $\mathbf{k}$. Moreover, any two such $(E, H; \Psi)$ are isomorphic.*

Proof. There is a complex vector bundle E_1 over Σ together with an isomorphism $\eta\colon E_0|_{\Sigma\backslash\{z_0\}} \cong E_1|_{\Sigma\backslash\{z_0\}}$ which can be written as $\operatorname{diag}(z^{k_1}, \dots, z^{k_r})$ in suitable trivializations around z_0. One can construct E_1 and η, for example, by modifying a Čech cocycle representing E_0. The complex vector bundle E is now constructed by gluing via η the pullback of E_0 to $[0, y_0] \times \Sigma\backslash\{(y_0, z_0)\}$ and the pullback of E_1 to $[y_0, 1] \times \Sigma\backslash\{(y_0, z_0)\}$. Since E is isomorphic near (y_0, z_0) to $E_{\mathbf{k}}$, we can find the desired Hermitian metric H and framing Ψ. ■

Henceforth, we fix a choice of

$$(E, H; \Psi).$$

Definition 4.2. Denote by $\mathscr{C}^{\mathrm{EBE}}(A_0, \phi_0; y_0, z_0, \mathbf{k})$ the set of triples $A \in \mathscr{A}(E, H)$, $\phi \in \Omega^1(M, \mathfrak{u}(E, H))$, and $\xi \in \Omega^0(M, \mathfrak{u}(E, H))$ satisfying the extended Bogomolny equation (3.2), as well as

$$i(\partial_y)\phi = 0, \tag{4.3}$$

and the boundary conditions

$$A|_{\{0\}\times\Sigma} = A_0, \quad \phi|_{\{0\}\times\Sigma} = \phi_0, \quad \text{and} \quad \xi|_{\{1\}\times\Sigma} = 0. \tag{4.4}$$

Denote by

$$\mathscr{G} \subset \mathscr{G}(E, H)$$

the subgroup of singularity preserving unitary gauge transformations of (E, H) which restrict to the identity on $\{0\} \times \Sigma$. Set

$$\mathscr{M}^{\mathrm{EBE}}(A_0, \phi_0; y_0, z_0, \mathbf{k}) \coloneqq \mathscr{C}^{\mathrm{EBE}}(A_0, \phi_0; y_0, z_0, \mathbf{k})/\mathscr{G}.$$ •

Remark 4.5. It is an interesting question to ask whether the condition (4.3) really does need to be imposed. In a variant of our setup on $S^1 \times \Sigma$, this condition is automatically satisfied; see [He17, Corollary 4.7]. ♣

Remark 4.6. We refer the reader to [KW07, Section 10.1] for a discussion of the significance of the boundary conditions (4.4). It will become apparent in Section 7 and (9.2), that the boundary conditions on (A, φ, ξ) correspond to Dirichlet and Neumann boundary conditions on a Hermitian metric. ♣

Proposition 4.7. *Let $A \in \mathscr{A}(E, H)$, $\phi \in \Omega^1(M, \mathfrak{u}(E, H))$, and $\xi \in \Omega^0(M, \mathfrak{u}(E, H))$ and suppose that (4.3) holds. Decompose A as*

$$\nabla_A = \partial_A + \bar{\partial}_A + \mathrm{d}y \wedge \nabla_{A,\partial_y}$$

and write

$$\phi = \varphi - \varphi^* \quad \textit{with} \quad \varphi := \frac{1}{2}(\phi - iI\phi) \in \Gamma(\pi_\Sigma^* T^*\Sigma^{1,0} \otimes \mathrm{End}(E)).^{1}$$

Set

$$\mathfrak{d}_y := \nabla_{A,\partial_y} - i\xi.$$

The extended Bogomolny equation (3.2) holds if and only if

$$\bar{\partial}_A\varphi = 0, \quad [\mathfrak{d}_y, \bar{\partial}_A] = 0, \quad \mathfrak{d}_y\varphi = 0, \quad \textit{and} \tag{4.8}$$

$$i\Lambda(F_A + [\varphi \wedge \varphi^*]) - i\nabla_{A,\partial_y}\xi = 0. \tag{4.9}$$

Proof. By the Kähler identities,

$$\mathrm{d}_A^*\phi = i\Lambda(\bar{\partial}_A\varphi + \partial_A\varphi^*).$$

Since $*_\Sigma = -I$, $*_\Sigma\varphi = i\varphi$ and thus

$$*\varphi = i\mathrm{d}y \wedge \varphi.$$

Therefore, the second equation of (3.2) is equivalent to

$$\begin{aligned} \bar{\partial}_A\varphi - \partial_A\varphi^* &= 0, \\ \nabla_{A,\partial_y}\varphi - i[\xi, \varphi] &= 0, \quad \text{and} \\ \nabla_{A,\partial_y}\varphi^* + i[\xi, \varphi^*] &= 0. \end{aligned}$$

This shows that the last two equations of (3.2) are equivalent to the first and the last equations of (4.8).

We have

$$\begin{aligned} F_A &= \bar{\partial}_A\partial_A + \partial_A\bar{\partial}_A + \mathrm{d}y \wedge \Big([\nabla_{A,\partial_y}, \bar{\partial}_A] + [\nabla_{A,\partial_y}, \partial_A]\Big), \\ \frac{1}{2}[\phi \wedge \phi] &= -[\varphi \wedge \varphi^*], \quad \text{and} \\ *\mathrm{d}_A\xi &= \nabla_{A,\partial_y}\xi \cdot \mathrm{vol}_\Sigma + i\mathrm{d}y \wedge \partial_A\xi - i\mathrm{d}y \wedge \bar{\partial}_A\xi. \end{aligned}$$

[1] This is possible because of (4.3).

Therefore, the first equation of (3.2) is equivalent to

$$\begin{aligned}\bar\partial_A\partial_A + \partial_A\bar\partial_A + [\varphi\wedge\varphi^*] - \nabla_{A,\partial_y}\xi\cdot\mathrm{vol}_\Sigma &= 0,\\ [\nabla_{A,\partial_y},\partial_A] - i\partial_A\xi &= 0, \quad\text{and}\\ [\nabla_{A,\partial_y},\bar\partial_A] + i\bar\partial_A\xi &= 0.\end{aligned}$$

These are precisely the second equation in (4.8) as well as (4.9). ■

5 The scattering map

Definition 5.1. In the situation of Example 3.4, set

$$\bar\partial_{\mathbf{k}} \coloneqq \bar\partial_{A_{\mathbf{k}}} \quad\text{and}\quad \mathfrak{d}_{y,\mathbf{k}} \coloneqq \nabla_{A_{\mathbf{k}},\partial_y} - i\xi_{\mathbf{k}}. \qquad \bullet$$

Definition 5.2. A **parametrized Hecke modification** on $(E, H; \Psi)$ is a triple $(\bar\partial, \varphi, \mathfrak{d}_y)$ consisting of:

(1) a complex linear map $\bar\partial\colon \Gamma(E) \to \Gamma(\mathrm{Hom}(\pi_\Sigma^* T\Sigma^{0,1}, E))$,

(2) a section $\varphi \in \Gamma(\pi_\Sigma^* T^*\Sigma^{1,0} \otimes \mathrm{End}(E))$, and

(3) a complex linear map $\mathfrak{d}_y\colon \Gamma(E) \to \Gamma(E)$

such that the following hold:

(4) For every $s \in \Gamma(E)$ and $f \in C^\infty(M, \mathbf{C})$

$$\bar\partial(fs) = (\bar\partial_\Sigma f)\otimes s + f\bar\partial s \quad\text{and}\quad \mathfrak{d}_y(fs) = (\partial_y f)s + f\mathfrak{d}_y s.$$

(5) There exists an $\alpha > 0$ such that for every $k \in \mathbf{N}_0$

$$\begin{gathered}\nabla^k_{A_{\mathbf{k}}}(\Psi_*\bar\partial - \bar\partial_{\mathbf{k}}) = O(r^{-k-1+\alpha}), \quad \nabla^k_{A_{\mathbf{k}}}\Psi_*\varphi = O(r^{-k}), \quad\text{and}\\ \nabla^k_{A_{\mathbf{k}}}(\Psi_*\mathfrak{d}_y - \mathfrak{d}_y^{\mathbf{k}}) = O(r^{-k-1+\alpha}).\end{gathered} \tag{5.3}$$

(6) We have

$$\bar\partial\varphi = 0, \quad [\mathfrak{d}_y, \bar\partial] = 0, \quad\text{and}\quad [\mathfrak{d}_y, \varphi] = 0. \tag{5.4}$$ •

The following observation is fundamental to this article.

Proposition 5.5 (Kapustin and Witten [KW07, Section 9.1]). *Let $(\bar\partial, \varphi, \mathfrak{d}_y)$ be a parametrized Hecke modification. Denote by $(\mathscr{E}_0, \varphi_0)$ and $(\mathscr{E}_1, \varphi_1)$ the Higgs bundles induced by restriction to $\{0\}\times\Sigma$ and $\{1\}\times\Sigma$ respectively. The parallel transport associated with the operator $\mathfrak{d}_y$ induces a Hecke modification*

$$\sigma\colon (\mathscr{E}_0, \varphi_0)|_{\Sigma\setminus\{z_0\}} \to (\mathscr{E}_1, \varphi_1)|_{\Sigma\setminus\{z_0\}}$$

at z_0 of type $\mathbf{k}$.

Definition 5.6. We call σ the **scattering map** associated with $(\bar\partial, \varphi, \mathfrak{d}_y)$. •

For the reader's convenience we recall the proof of Proposition 5.5 following [CH11].

Proposition 5.7 (Charbonneau and Hurtubise [CH11, Section 2.2]). *The scattering map for the Dirac monopole of type* $\mathbf{k}$ *is given by* $\operatorname{diag}(z^{k_1}, \dots, z^{k_r})$ *in suitable holomorphic trivializations.*

Proof. It suffices to consider the case $r = 1$. Set

$$U_\pm \coloneqq \{(y,z) \in \mathbf{R} \times \mathbf{C} : z = 0 \implies \pm y > 0\}.$$

There are trivializations $\tau_\pm \colon \pi^*\mathscr{O}_{\mathbf{C}P^1}(k)|_{U_\pm} \cong U_\pm \times \mathbf{C}$ such that the following hold:

(1) The transition function $\tau\colon U_+ \cap U_- \to \mathrm{U}(1)$ defined by

$$\tau_+ \circ \tau_-^{-1}(y,z;\lambda) =\colon (y,z,\tau(y,z)\lambda)$$

is given by

$$(y,z) \mapsto (z/|z|)^k.$$

(2) The connection A defined in Example 3.4 satisfies

$$\nabla_{A_\pm} \coloneqq (\tau_\pm)_*\nabla_A = \mathrm{d} + \frac{k}{4}(\mp 1 + y/r)\frac{\bar z \mathrm{d}z - z\mathrm{d}\bar z}{|z|^2}$$

for

$$r \coloneqq \sqrt{y^2 + |z|^2}.$$

The trivializations $\tau_\pm$ are not holomorphic. This can be rectified as follows. Since

$$\mathrm{d}r = \frac{1}{2r}(\bar z\mathrm{d}z + z\mathrm{d}\bar z + 2y\mathrm{d}y),$$

the gauge transformations

$$u_\pm(y,z) \coloneqq (r \pm y)^{\pm k/2}$$

satisfy

$$\begin{aligned}
-(\mathrm{d}u_\pm)u_\pm^{-1} &= \mp\frac{k}{2(r\pm y)}(\mathrm{d}r \pm \mathrm{d}y) \\
&= \mp\frac{k}{4r(r\pm y)}(\bar z\mathrm{d}z + z\mathrm{d}\bar z + 2(y \pm r)\mathrm{d}y) \\
&= \frac{k}{4}(\mp 1 + y/r)\frac{\bar z\mathrm{d}z + z\mathrm{d}\bar z}{|z|^2} - \frac{k}{2r}\mathrm{d}y.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\nabla_{\tilde A_\pm} &\coloneqq (u_\pm)_*\nabla_{A_\pm} \\
&= \nabla_{A_\pm} - (\mathrm{d}u_\pm)u_\pm^{-1} \\
&= \mathrm{d} + \frac{k}{2}(\mp 1 + y/r)\frac{\bar z\mathrm{d}z}{|z|^2} - \frac{k}{2r}\mathrm{d}y.
\end{aligned}$$

It follows that

$$\bar\partial_{\tilde A_\pm} = \bar\partial \quad \text{and} \quad \nabla_{\tilde A_\pm, \partial_y} + \frac{k}{2r} = \partial_y.$$

Hence, the trivializations $u_\pm \circ \tau_\pm$ are holomorphic and with respect to these the parallel transport associated with $\nabla_{A,\partial_y} + \frac{ik}{2r}$ from $y = -\varepsilon$ to $y = \varepsilon$ is given by

$$u_+(\varepsilon, z) \cdot \tau(\varepsilon, z) \cdot u_-^{-1}(-\varepsilon, z) = (r+\varepsilon)^{k/2} \left(\frac{z}{|z|}\right)^k (r-\varepsilon)^{k/2} = z^k. \qquad \blacksquare$$

Proof of Proposition 5.5. The fact that σ is holomorphic and preserves the Higgs fields follows directly from (5.4).

To prove that σ is given by $\operatorname{diag}(z^{k_1}, \dots, z^{k_r})$ in suitable trivializations we follow Charbonneau and Hurtubise [CH11, Proposition 2.5]. It suffices to consider a neighborhood of (y_0, z_0) which we identify with a neighborhood of the origin in $\mathbf{R} \times \mathbf{C}$. Since $\mathfrak{d}_y = \mathfrak{d}_{y,\mathbf{k}} + O(r^{-1+\alpha})$, we can construct a section τ of $\operatorname{End}(E_\mathbf{k})$ over $[-\varepsilon, 0) \times \{0\}$ satisfying

$$\mathfrak{d}_y \tau = \tau \mathfrak{d}_{y,\mathbf{k}} \quad \text{and} \quad \tau(\cdot, 0) = \mathrm{id}_{\mathbf{C}^r} + O(r^\alpha). \tag{5.8}$$

First extend $\tau(-\varepsilon, 0)$ to a section of $\operatorname{End}(E_\mathbf{k})$ over $\{-\varepsilon\} \times B_\varepsilon(0)$ satisfying

$$\bar\partial \tau = \tau \bar\partial_\mathbf{k} \tag{5.9}$$

and then further extend it to $[-\varepsilon, \varepsilon] \times B_\varepsilon(0) \backslash [0, \varepsilon] \times \{0\}$ by imposing the first part of (5.8). The equation (5.9) continues to hold. Since τ is bounded around $(0,0)$, it extends to $[-\varepsilon, \varepsilon] \times B_\varepsilon(0)$. If $0 < \varepsilon \ll 1$, then τ is invertible.

By construction, if σ denotes the parallel transport associated with $\mathfrak{d}_{y,\mathbf{k}}$ from $y = -\varepsilon$ to $y = \varepsilon$, then the corresponding parallel transport associated with $\mathfrak{d}_y$ is given by

$$\tau(\varepsilon, \cdot) \sigma \tau(-\varepsilon, \cdot)^{-1}. \qquad \blacksquare$$

In light of Proposition 5.7, this proves the assertion.

The preceding discussion constructs a map

$$\mathscr{C}^{\mathrm{EBE}}(A_0, \phi_0; y_0, z_0, \mathbf{k}) \to \mathscr{M}^{\mathrm{Hecke}}(\mathscr{E}_0, \varphi_0; z_0, \mathbf{k}).$$

This map is $\mathscr{G}$–invariant. The following is the main result of this article.

Theorem 5.10. *The map*

$$\mathscr{M}^{EBE}(A_0, \phi_0; y_0, z_0, \mathbf{k}) \to \mathscr{M}^{Hecke}(\mathscr{E}_0, \varphi_0; z_0, \mathbf{k})$$

induced by the scattering map construction is bijective.

The proof of this theorem occupies the remainder of this article.

6 Parametrizing Hecke modifications

Definition 6.1. Denote by $(\mathscr{E}_0, \varphi_0)$ the Higgs bundle induced by (A_0, ϕ_0). Denote by

$$\widetilde{\mathscr{C}^{\mathrm{Hecke}}}(\mathscr{E}_0, \varphi_0; y_0, z_0, \mathbf{k})$$

the set of parametrized Hecke modifications agreeing with $(\mathscr{E}_0, \varphi_0)$ at $y = 0$. Denote by

$$\mathscr{G}^{\mathbf{C}} \subset \mathscr{G}^{\mathbf{C}}(E)$$

the group of singularity preserving complex gauge transformations of E which are the identity at $y = 0$. Here singularity preserving means the analogue of the condition in Definition 3.6 holds.

Set

$$\widetilde{\mathscr{M}^{\mathrm{Hecke}}}(\mathscr{E}_0, \varphi_0; y_0, z_0, \mathbf{k}) \coloneqq \widetilde{\mathscr{C}^{\mathrm{Hecke}}}(\mathscr{E}_0, \varphi_0; y_0, z_0, \mathbf{k})/\mathscr{G}^{\mathbf{C}}. \qquad \bullet$$

The first step in the proof of Theorem 5.10 is to show that every Hecke modification of $(\mathscr{E}_0, \varphi_0)$ arises as the scattering map of a parametrized Hecke modification.

Proposition 6.2. *The map*

$$\widetilde{\mathscr{M}^{Hecke}}(\mathscr{E}_0, \varphi_0; y_0, z_0, \mathbf{k}) \to \mathscr{M}^{Hecke}(\mathscr{E}_0, \varphi_0; z_0, \mathbf{k}) \tag{6.3}$$

induced by the scattering map construction is a bijection.

Proof. Let $(\mathscr{E}_1, \varphi_1; \eta)$ be a Hecke modification of $(\mathscr{E}_0, \varphi_0)$ at z_0 of type $\mathbf{k}$. Denote the complex vector bundles underlying $\mathscr{E}_0$ and $\mathscr{E}_1$ by E_0 and E_1. Denote the holomorphic structures on $\mathscr{E}_0$ and $\mathscr{E}_1$ by $\bar\partial_0$ and $\bar\partial_1$. The bundle E is isomorphic to the bundle obtained by gluing the pullback of E_0 to $[0, y_0] \times \Sigma \backslash \{(y_0, z_0)\}$ and the pullback of E_1 to $[y_0, 1] \times \Sigma \backslash \{(y_0, z_0)\}$ via η. Therefore, there is an operator $\bar\partial\colon \Gamma(E) \to \Gamma(\mathrm{Hom}(\pi_\Sigma^* T\Sigma^{0,1}, E))$ on E whose restriction to $\{y\} \times \Sigma$ agrees with $\bar\partial_0$ if $y < y_0$ and with $\bar\partial_1$ if $y > y_0$. There also is a section $\varphi \in \Gamma(\pi_\Sigma^* T^*\Sigma^{1,0} \otimes \mathrm{End}(E))$ whose restriction to $\{y\} \times \Sigma$ agrees φ_0 if $y < y_0$ and with φ_1 if $y > y_0$. Define $\mathfrak{d}_y\colon \Gamma(E) \to \Gamma(E)$ to be given by ∂_y on both halves of the above decomposition of E. By construction, $(\bar\partial, \varphi, \mathfrak{d}_y)$ is a parametrized Hecke modification and the associated scattering map induces the Hecke modification $(\mathscr{E}_1, \varphi_1; \eta)$. This proves that the map (6.3) is surjective.

Let $(\bar\partial, \varphi, \mathfrak{d}_y)$ and $(\tilde{\bar\partial}, \tilde\varphi, \tilde{\mathfrak{d}}_y)$ be two parametrized Hecke modification which induce the Hecke modifications $(\mathscr{E}_1, \varphi_1; \eta)$ and $(\tilde{\mathscr{E}}_1, \tilde\varphi_1; \tilde\eta)$. Suppose that the latter are isomorphic via $\zeta\colon (\mathscr{E}_1, \varphi_1) \to (\tilde{\mathscr{E}}_1, \tilde\varphi_1)$. We can assume that both parametrized Hecke modifications are in temporal gauge. Therefore, on $[0, y_0) \times \Sigma$ they agree and are given by $(\bar\partial_0, \varphi_0, \partial_y)$; while on $(y_0, 1] \times \Sigma$

$$(\bar\partial, \varphi, \mathfrak{d}_y) = (\bar\partial_1, \varphi_1, \partial_y) \quad \text{and} \quad (\tilde{\bar\partial}, \tilde\varphi, \tilde{\mathfrak{d}}_y) = (\tilde{\bar\partial}_1, \tilde\varphi_1, \partial_y).$$

The isomorphism ζ intertwines $\bar\partial_1$ and $\tilde{\bar\partial}_1$ as well as φ_1 and $\tilde\varphi_1$ and commutes with the identification of E_0 and E_1 respectively $\tilde E_1$ over $\Sigma \backslash \{z_0\}$. Therefore, it glues with the identity on E_0 to a gauge transformation in $\mathscr{G}^{\mathbf{C}}$ relating $(\bar\partial, \varphi, \mathfrak{d}_y)$ and $(\tilde{\bar\partial}, \tilde\varphi, \tilde{\mathfrak{d}}_y)$. This proves that the map (6.3) is injective. ■

7 Varying the Hermitian metric

The purpose of this section is to reduce Theorem 5.10 to a uniqueness and existence result for a certain partial differential equation imposed on a Hermitian metric.

Proposition 7.1. *Given a parametrized Hecke modification* $(\bar\partial, \varphi, \mathfrak{d}_y)$ *on* (E, H)*, there are unique* $A_H \in \mathscr{A}(E, H)$*,* $\phi_H \in \Omega^1(M, \mathfrak{u}(E, H))$*, and* $\xi_H \in \Omega^0(M, \mathfrak{u}(E, H))$ *such that*

$$\bar\partial = \nabla^{0,1}_{A_H}, \quad \varphi = \phi^{1,0}_H, \quad \textit{and} \quad \mathfrak{d}_y = \nabla_{A_H, \partial_y} - i\xi_H. \tag{7.2}$$

Moreover, (A_H, ϕ_H, ξ_H) *has a Dirac type singularity of type* $\mathbf{k}$ *at* (y_0, z_0).

Proof. This is analogous to the existence and uniqueness of the Chern connection. In fact, it can be reduced to it; see Proposition 8.1. ∎

This proposition shows that Theorem 5.10 is equivalent to the bijectivity of the map

$$\Big\{(\bar\partial, \varphi, \mathfrak{d}_y) \in \widetilde{\mathscr{C}^{\mathrm{Hecke}}}(\mathscr{E}_0, \varphi_0; y_0, z_0, \mathbf{k}) : (4.9) \text{ and } \xi_H(1, \cdot) = 0\Big\}/\mathscr{G} \to \widetilde{\mathscr{M}^{\mathrm{Hecke}}}(\mathscr{E}_0, \varphi_0; y_0, z_0, \mathbf{k}).$$

This in turn is equivalent to the following for every parametrized Hecke modification $(\bar\partial, \varphi, \mathfrak{d}_y)$:

(1) There exists a $u \in \mathscr{G}^{\mathbf{C}}$ such that $u_*(\bar\partial, \varphi, \mathfrak{d}_y)$ satisfies (4.9) and $\xi_H(1, \cdot) = 0$.

(2) The equivalence class $[u] \in \mathscr{G}^{\mathbf{C}}/\mathscr{G}$ is unique.

The gauge transformed parametrized Hecke modification $u_*(\bar\partial, \varphi, \mathfrak{d}_y)$ satisfies (4.9) and $\xi_H(1, \cdot) = 0$ if and only if with respect to gauge transformed Hermitian metric

$$K \coloneqq u_* H$$

the parametrized Hecke modification $(\bar\partial, \varphi, \mathfrak{d}_y)$ satisfies (4.9) and $\xi_K(1, \cdot) = 0$. Since $K = u_* H$ depends only on $[u] \in \mathscr{G}^{\mathbf{C}}/\mathscr{G}$, the preceding discussion shows that Theorem 5.10 holds assuming the following.

Proposition 7.3. *Given* $(\bar\partial, \varphi, \mathfrak{d}_y)$ *a parametrized Hecke modification, there exists a unique Hermitian metric of the form* $K = u_* H$ *with* $u \in \mathscr{G}^{\mathbf{C}}$ *such that* (4.9) *and* $\xi_K(1, \cdot) = 0$ *hold.*

8 Lift to dimension four

It will be convenient to lift the extended Bogomolny equation to dimension four, since this allows us to directly make use of the work of Simpson [Sim88].

Proposition 8.1. *Set*

$$X \coloneqq S^1 \times M.$$

Denote by α *the coordinate on* S^1*. Regard* X *as a Kähler manifold equipped with the product metric and the Kähler form*

$$\omega = \mathrm{d}\alpha \wedge \mathrm{d}y + \mathrm{vol}_\Sigma.$$

Denote by $\mathbf{E}$ *the pullback of* E *to* X*. Given a parametrized Hecke modification* $(\bar\partial, \varphi, \mathfrak{d}_y)$*, set*

$$\bar{\boldsymbol{\partial}} := \frac{1}{2}(\partial_\alpha + i\mathrm{d}y \cdot \mathfrak{d}_y) + \bar\partial_E \quad \textit{and} \quad \boldsymbol{\varphi} := \varphi.$$

The following hold:

(1) *The operator* $\bar{\boldsymbol{\partial}}$ *defines a holomorphic structure on* $\mathbf{E}$*; moreover,*

$$\bar{\boldsymbol{\partial}}\boldsymbol{\varphi} = 0 \quad \textit{and} \quad \boldsymbol{\varphi} \wedge \boldsymbol{\varphi} = 0.$$

(2) *Let* $\mathbf{K}$ *be the pullback of a Hermitian metric* K *on* E*. Denote by* $A_{\mathbf{K}}$ *the Chern connection corresponding to* $\bar{\boldsymbol{\partial}}$ *with respect to* $\mathbf{K}$*. The equation (4.9) holds if and only if*

$$i\Lambda(F_{A_{\mathbf{K}}} + [\boldsymbol{\varphi} \wedge \boldsymbol{\varphi}^{*,\mathbf{K}}]) = 0.$$

Proof. It follows from (4.8) that

$$\bar{\boldsymbol{\partial}}^2 = \bar\partial_E^2 + i\mathrm{d}y \wedge [\mathfrak{d}_y, \bar\partial_E] = 0.$$

Consequently, $\bar{\boldsymbol{\partial}}$ defines a holomorphic structure. It also follows from (4.8) that $\bar{\boldsymbol{\partial}}\boldsymbol{\varphi} = 0$; while $\boldsymbol{\varphi} \wedge \boldsymbol{\varphi} = 0$ is obvious. This proves (1).

Denote by $\pi\colon X \longrightarrow M$ the projection map. A computation shows that

$$A_{\mathbf{K}} = \pi^* A_K + \mathrm{d}\alpha \wedge (\partial_\alpha + \xi_K).$$

Therefore,

$$F_{A_{\mathbf{K}}} = F_{A_K} - \mathrm{d}\alpha \wedge \mathrm{d}y \cdot \nabla_{A_K, \partial_y} \xi_K$$

and thus

$$i\Lambda(F_{A_{\mathbf{K}}} + [\boldsymbol{\varphi} \wedge \boldsymbol{\varphi}^{*,\mathbf{K}}]) = \pi^* \left[i\Lambda(F_{A_K} + [\varphi \wedge \varphi^{*,K}]) - i\nabla_{A_K,\partial_y}\xi_K \right].$$

This proves (2). ■

9 Uniqueness of K

Assume the situation of Proposition 7.3. Given a Hermitian metric K on E, set

$$\mathfrak{m}(K) := i\Lambda(F_{A_K} + [\varphi \wedge \varphi^{*,K}]) - i\nabla_{A_K,\partial_y}\xi_K.$$

Thus, (4.9) holds with respect to K if and only if $\mathfrak{m}(K) = 0$.

Proposition 9.1. *For every Hermitian metric* K *on* E *and* $s \in \Gamma(i\mathfrak{u}(E, K))$*,*

$$\Delta \operatorname{tr} s = 2 \operatorname{tr}(\mathfrak{m}(Ke^s) - \mathfrak{m}(K))$$

and

$$\Delta \log \operatorname{tr} e^s \leqslant 2|\mathfrak{m}(Ke^s)| + 2|\mathfrak{m}(K)|$$

Furthermore, if s *is trace-free, then* $\mathfrak{m}(Ke^s)$ *and* $\mathfrak{m}(K)$ *can be replaced by their trace-free parts.*

Proof. This follows from [Sim88, Lemma 3.1(c) and (d)] and Proposition 8.1. ∎

Proof of uniqueness in Proposition 7.3. Suppose K and Ke^s are two Hermitian metrics in the $\mathscr{G}^{\mathbf{C}}$–orbit of H such that $\mathfrak{m}(K) = \mathfrak{m}(Ke^s) = 0$ and $\xi_K(1,\cdot) = \xi_{Ke^s}(1,\cdot) = 0$. It follows from the preceding proposition that $\operatorname{tr} s$ is harmonic and $\log \operatorname{tr} e^s$ is subharmonic.

Since K and Ke^s are contained the the same $\mathscr{G}^{\mathbf{C}}$–orbit,

$$s(0,\cdot) = 0 \quad \text{and} \quad |s| = O(r^\alpha).$$

for some $\alpha > 0$. The computation proving Proposition 7.1 shows that

$$\xi_{Ke^s} = \frac{1}{2}\left(\xi_K + e^{-s}\xi_K e^s - ie^{-s}(\nabla_{A_K,\partial_y} e^s)\right). \tag{9.2}$$

Therefore,

$$\nabla_{A_K,\partial_y} s(1,\cdot) = 0.$$

Since $\operatorname{tr} s$ is harmonic, bounded, vanishes at $y = 0$, and satisfies Neumann boundary conditions at $y = 1$, it follows that $\operatorname{tr} s = 0$. Furthermore, since $\log \operatorname{tr} e^s$ is subharmonic, the above together with the maximum principle implies $\log \operatorname{tr} e^s \leqslant \log \operatorname{tr} e^0 = \log \operatorname{rk} E$. By the inequality between arithmetic and geometric means,

$$\frac{\operatorname{tr} e^s}{\operatorname{rk} E} \geqslant e^{\operatorname{tr} s} = 1; \quad \text{that is:} \quad \log \operatorname{tr} e^s \geqslant \log \operatorname{rk} E$$

with equality if and only if $s = 0$. ∎

10 Construction of K

This section is devoted to the construction of K using the heat flow method with boundary conditions [Sim88; Don92]. The analysis of its behavior at the singularity is discussed in the next section.

Proposition 10.1. *Given a parametrized Hecke modification, $(\bar\partial, \varphi, \mathfrak{d}_y)$ on (E, H), there exists a bounded section $s \in \Gamma(i\mathfrak{u}(E,H))$ such that for $K := He^s$ both $\mathfrak{m}(K) = 0$ and $\xi_K(1,\cdot) = 0$ hold.*

The proof requires the following result as a preparation.

Proposition 10.2. *Assume the situation of Proposition 8.1. For $\varepsilon > 0$, set*

$$X_\varepsilon := S^1 \times ([0,1] \times \Sigma \backslash B_\varepsilon(y_0, z_0)).$$

Denote the pullback of H to X by $\mathbf{H}$. Suppose that

$$\|i\Lambda(F^\circ_{A_{\mathbf{H}}} + [\boldsymbol{\varphi} \wedge \boldsymbol{\varphi}^{*,\mathbf{H}}])\|_{L^\infty} < \infty.$$

The following hold:

(1) *Let $\varepsilon > 0$. There exists a unique solution $(\mathbf{K}_t^\varepsilon)_{t\in[0,\infty)}$ of*

$$(\mathbf{K}_t^\varepsilon)^{-1}\partial_t\mathbf{K}_t^\varepsilon = -i\Lambda(F^\circ_{A_{\mathbf{K}_t^\varepsilon}} + [\boldsymbol{\varphi}\wedge\boldsymbol{\varphi}^{*,\mathbf{K}_t^\varepsilon}]) \tag{10.3}$$

on X_ε with initial condition

$$\mathbf{K}_0^\varepsilon = \mathbf{H}|_{X_\varepsilon}$$

and subject to the boundary conditions

$$\begin{aligned}\mathbf{K}_t^\varepsilon|_{S^1\times\{0\}\times\Sigma} &= \mathbf{H}|_{S^1\times\{0\}\times\Sigma},\\ \mathbf{K}_t^\varepsilon|_{S^1\times\partial B_\varepsilon(y_0,z_0)} &= \mathbf{H}|_{S^1\times\partial B_\varepsilon(y_0,z_0)}, \quad \textit{and}\\ (\nabla_{A_{\mathbf{H}},\partial_y}\mathbf{K}_t^\varepsilon)|_{S^1\times\{1\}\times\Sigma} &= 0.\end{aligned}$$

(2) *As $t\to\infty$, the Hermitian metrics $\mathbf{K}_t^\varepsilon$ converge in C^∞ to a solution $\mathbf{K}^\varepsilon$ of*

$$i\Lambda(F^\circ_{\mathbf{K}^\varepsilon} + [\boldsymbol{\varphi}\wedge\boldsymbol{\varphi}^{*,\mathbf{K}^\varepsilon}]) = 0.$$

(3) *The section $s_\varepsilon \in \Gamma(X_\varepsilon, i\mathfrak{su}(\mathbf{E},\mathbf{H}))$ defined by $\mathbf{K}^\varepsilon = \mathbf{H}e^{s_\varepsilon}$ is S^1−invariant and satisfies*

$$\|s_\varepsilon\|_{L^\infty} \lesssim 1 \quad \textit{as well as} \quad \|s_\varepsilon\|_{C^k(X_\delta)} \lesssim_{k,\delta} 1$$

for every $k\in\mathbf{N}$ and $\delta > \varepsilon$.

Proof. (1) follows from Simpson [Sim88, Section 6].

Set

$$f_t := |i\Lambda(F^\circ_{\mathbf{K}_t} + [\boldsymbol{\varphi}\wedge\boldsymbol{\varphi}^*])|^2_{\mathbf{K}_t}.$$

By a short computation, we have

$$(\partial_t + \Delta)f_t \leqslant 0.$$

The spectrum of Δ on X_ε with Dirichlet boundary conditions at $y = 0$ and at distance ε to the singularity as well as Neumann boundary conditions at $y = 0$ is positive. Therefore, there are $c, \lambda > 0$ such that

$$\|f_t\|_{L^\infty} \leqslant ce^{-\lambda t}.$$

Consequently,

$$\sup_{p\in X_\varepsilon}\int_0^\infty \sqrt{f_t}\mathrm{d}t < \infty$$

This means that the path $\mathbf{K}_t^\varepsilon$ has finite length in the space of Hermitian metrics. (2) thus follows from [Sim88, Lemma 6.4]. The S^1−invariance of s_ε follows from the S^1−invariance of the initial condition.from [Sim88, Theorem 1].

Since s_ε is S^1−invariant and trace-free, by Proposition 8.1 and Proposition 9.1,

$$\Delta\log\operatorname{tr}(e^{s_\varepsilon}) \leqslant 2|i\Lambda(F^\circ_{A_{\mathbf{H}}} + [\boldsymbol{\varphi}\wedge\boldsymbol{\varphi}^{*,\mathbf{H}}])|^2.$$

Let f be the solution of

$$\Delta f = 2|i\Lambda(F^\circ_{A_{\mathbf{H}}} + [\boldsymbol{\varphi}\wedge\boldsymbol{\varphi}^{*,\mathbf{H}}])|^2$$

subject to the boundary conditions

$$f|_{S^1\times\{0\}\times\Sigma} = 0 \quad \text{and} \quad \partial_y f|_{S^1\times\{1\}\times\Sigma} = 0.$$

Choose a constant c such that $f + c > 0$. Set

$$g := \log \operatorname{tr}(e^{s_\varepsilon}) - (f + c).$$

The function g is subharmonic on X_ε. Thus it achieves its maximum on the boundary. On $S^1 \times \partial B_\varepsilon(y_0, z_0)$ and $S^1 \times \{0\} \times \Sigma$, the function g is negative. At $S^1 \times \{1\} \times \Sigma$, $\partial_y f = 0$. By the reflection principle, the maximum is not achieved at $y = 1$ unless g is constant. It follows that $g \leqslant 0$. This shows that $|\log \operatorname{tr}(e^{s_\varepsilon})|$ is bounded independent of ε. Since s is trace-free, it follows that $|s_\varepsilon|$ is bounded independent of ε. By [Sim88, Lemma 6.4], which is an extension of [Don85, Lemma 19] with boundary conditions, and elliptic bootstrapping the asserted C^k bounds on s_ε follow. ■

Proof of Proposition 10.1. Without loss of generality we can assume that H is such that ξ_H vanishes at $y = 1$.

There is a unique $f \in C^\infty([0, 1] \times \Sigma\backslash\{y_0, z_0\})$ which satisfies

$$\frac{1}{2}\Delta f = \operatorname{tr}(i\Lambda F_{A_H} - i\nabla_{A_H,\partial_y}\xi_H),$$

is bounded, vanishes at $y = 0$, and satisfies Neumann boundary conditions at $y = 0$. A barrier argument shows that $|f| = O(r^\alpha)$ for some $\alpha > 0$. Replacing H with He^f, we may assume that

$$\operatorname{tr}(i\Lambda F_{A_H} - i\nabla_{A_H,\partial_y}\xi_H) = 0.$$

For every $s \in \Gamma(i\mathfrak{su}(E, H))$, the above condition holds for He^s instead of H as well. Let s_ε be as in Proposition 10.2. Take the limit of s_ε on each X_δ as first ε tends to zero and then δ tends to zero. This limit is the pullback of a section s defined over $[0, 1] \times \Sigma\backslash\{y_0, z_0\}$ which has the desired properties. Since $\nabla_{A_H,\partial_y} s$ vanishes at $y = 1$, it follows from (9.2) that ξ_K vanishes at $y = 1$. ■

11 Singularity analysis

It remains to analyze the section s constructed via Proposition 10.1 near the singularity. The following result completes the proof of Proposition 7.3 and thus Theorem 5.10.

Proposition 11.1. *Consider the unit ball $B \subset \mathbf{R} \times \mathbf{C}$ with a metric $g = g_0 + O(r^2)$. Set $\dot{B} := B\backslash\{0\}$. Let $\mathbf{k} \in \mathbf{Z}^r$ be such that (2.6) and let $\alpha > 0$. Let $(\bar\partial, \phi, \mathfrak{d}_y)$ be a parametrized Hecke modification on $(E_\mathbf{k}, H_\mathbf{k})$. If $s \in \Gamma(i\mathfrak{u}(E_\mathbf{k}, H_\mathbf{k}))$ is bounded and satisfies*

$$\mathfrak{m}(H_\mathbf{k} e^s) = 0,$$

then there is an $\alpha > 0$ and $s_0 \in \Gamma(i\mathfrak{u}(E_\mathbf{k}, H_\mathbf{k}))$ such that

$$\nabla_{A_\mathbf{k}} s_0 = 0 \quad \textit{and} \quad [\xi_\mathbf{k}, s_0] = 0$$

and for every $k \in \mathbf{N}_0$

$$\nabla^k_{A_\mathbf{k}}(s - s_0) = O(r^{-k+\alpha});$$

that is: $H_\mathbf{k} e^s = e_*^{s/2} H_\mathbf{k}$ *is in the* $\mathcal{G}^\mathbf{C}$*–orbit of* $H_\mathbf{k}$.

The proof of this result uses the technique developed in [JSW18]. Henceforth, we shall assume the situation of Proposition 11.1. Moreover, we drop the subscript $\mathbf{k}$ from $E_\mathbf{k}$ and $H_\mathbf{k}$ to simplify notation.

Define $\mathfrak{B} \colon \Gamma(i\mathfrak{u}(E,H)) \to \Omega^1(\dot{B}, i\mathfrak{u}(E,H)) \times \Gamma(i\mathfrak{u}(E,H))$ by

$$\mathfrak{B}s := (\nabla_{A_\mathbf{k}} s, [\xi_\mathbf{k}, s])$$

The following a priori Morrey estimate is the crucial ingredient of the proof of Proposition 11.1.

Proposition 11.2. *For some $\alpha > 0$, we have*

$$\int_{B_r} |\mathfrak{B}s|^2 \lesssim r^{1+2\alpha}.$$

Proof of Proposition 11.1 assuming Proposition 11.2. Denote by s_r the pullback of s from B_r to B. By Proposition 11.2,

$$\|\nabla_{A_\mathbf{k}} s_r\|_{L^2(B)} + \|[\xi_\mathbf{k}, s_r]\|_{L^2(B)} \lesssim r^\alpha.$$

Denote by $\mathfrak{m}_r$ the map $\mathfrak{m}$ with respect to r^{-2} times the pullback of the Riemannian metric and the parametrized Hecke modification from B_r to B. The equation $\mathfrak{m}_r(He^{s_r}) = 0$ can be written schematically as

$$\nabla^*_{A_H}\nabla_{A_H} s_r + B(\nabla_{A_H} s \otimes \nabla_{A_H} s_r) = C(\mathfrak{m}_r(H))$$

where B and C are linear with coefficients depending only on s, but not its derivatives.

Set

$$a := \nabla_{A_H} - \nabla_{A_\mathbf{k}}, \quad \hat{\phi} = \phi_H - \phi_\mathbf{k}, \quad \text{and} \quad \hat{\xi} := \xi_H - \xi_\mathbf{k}.$$

It follows from (5.3) that, after possibly decreasing the value of $\alpha > 0$, for $k \in \mathbf{N}_0$

$$\nabla^k_{A_\mathbf{k}} a = O(r^{-k-1+\alpha}), \quad \nabla^k_{A_\mathbf{k}} \hat{\phi} = O(r^{-k}), \quad \text{and} \quad \nabla^k_{A_\mathbf{k}} \hat{\xi} = O(r^{-k-1+\alpha}). \tag{11.3}$$

Therefore, $\mathfrak{m}_r(H) = O(r^\alpha)$ on $B \backslash B_{1/8}$.

As in [JSW18, Section 5], it follows from Bando–Siu's interior estimates [BS94, Proposition 1; JW19, Theorem C.1] that for $k \in \mathbf{N}_0$

$$\|\nabla_{A_\mathbf{k}} s_r\|_{C^k(B_{1/2}\backslash B_{1/4})} + \|[\xi_\mathbf{k}, s_r]\|_{C^k(B_{1/2}\backslash B_{1/4})} \lesssim_k r^\alpha.$$

Consequently, there is an $s_0 \in \ker \mathfrak{B}$ such that for $k \in \mathbf{N}_0$

$$\|\nabla^k_{A_\mathbf{k}}(s_r - s_0)\|_{L^\infty(B_{1/2}\backslash B_{1/4})} \lesssim_k r^\alpha.$$

This translates to the asserted estimates for s. ∎

The proof of Proposition 11.2 occupies the remainder of this section.

11.1 A Neumann–Poincaré inequality

Denoting the radial coordinate by r, we can write

$$\mathfrak{B}s := (\mathrm{d}r \cdot \nabla_{\partial_r} s, \mathfrak{B}_r s)$$

for a family of operators $\mathfrak{B}_r \colon \Gamma(\partial B_r, i\mathfrak{u}(E,H)) \to \Omega^1(\partial B_r, i\mathfrak{u}(E,H)) \times \Gamma(\partial B_r, i\mathfrak{u}(E,H))$. The pullback of $\mathfrak{B}_r$ to ∂B agrees with $\mathfrak{B}_1$. Consequently, we can identify

$$\ker \mathfrak{B}_r = \ker \mathfrak{B}_1 =: N.$$

Denote by $\pi_r \colon \Gamma(\partial B_r, i\mathfrak{u}(E,H)) \to N$ the L^2–orthogonal projection onto N. Set

$$\Pi_r s := \frac{1}{r} \int_r^{2r} \pi_t(s)\,\mathrm{d}t.$$

Proposition 11.4. *For every $s \in \Gamma(i\mathfrak{u}(E,H))$ and $r \in [0,1/2]$, we have*

$$\int_{B_{2r}\setminus B_r} |s - \Pi_r s|^2 \lesssim r^2 \int_{B_{2r}\setminus B_r} |\mathfrak{B}s|^2. \tag{11.5}$$

Proof. The proof is identical to that of [JSW18, Proposition 4.2]. For the readers convenience we will reproduce the argument here.

Since (11.5) is scale invariant, we may assume $r = 1/2$. Furthermore, it suffices to prove the cylindrical estimate

$$\int_{1/2}^1 \int_{\partial B} |s(t,\hat{x}) - \Pi s(t,\cdot)|^2\,\mathrm{d}\hat{x}\mathrm{d}t \lesssim \int_{1/2}^1 \int_{\partial B} |\partial_t s(t,\hat{x})|^2 + |\mathfrak{B}_1 s(t,\hat{x})|^2\,\mathrm{d}\hat{x}\mathrm{d}t$$

with s denoting a section over $[1/2,1] \times \partial B$,

$$\pi := \pi_1, \quad \text{and} \quad \Pi s := 2\int_{1/2}^1 \pi s(t,\cdot)\,\mathrm{d}t.$$

To prove this inequality, we compute

$$\begin{aligned}
&\int_{1/2}^1 \int_{\partial B} |s(t,\hat{x}) - \Pi s(t,\cdot)|^2 \mathrm{d}\hat{x}\mathrm{d}t \\
&\quad = 4\int_{1/2}^1 \int_{\partial B} \left| \int_{1/2}^1 s(t,\hat{x}) - \pi s(u,\cdot)\,\mathrm{d}u \right|^2 \mathrm{d}\hat{x}\mathrm{d}t \\
&\quad \lesssim \int_{1/2}^1 \int_{1/2}^1 \int_{\partial B} |s(t,\hat{x}) - \pi s(u,\cdot)|^2\,\mathrm{d}\hat{x}\mathrm{d}u\mathrm{d}t \\
&\quad \lesssim \int_{1/2}^1 \int_{1/2}^1 \int_{\partial B} |s(t,\hat{x}) - \pi s(t,\cdot)|^2 + |\pi s(t,\cdot) - \pi s(u,\cdot)|^2\,\mathrm{d}\hat{x}\mathrm{d}u\mathrm{d}t.
\end{aligned}$$

The first summand can be bounded as follows

$$\int_{1/2}^{1}\int_{1/2}^{1}\int_{\partial B} |s(t,\hat{x}) - \pi s(t,\cdot)|^2 \,\mathrm{d}\hat{x}\mathrm{d}t\mathrm{d}u \lesssim \int_{1/2}^{1}\int_{1/2}^{1}\int_{\partial B} |\mathfrak{B}_1 s(t,\hat{x})|^2 \,\mathrm{d}\hat{x}\mathrm{d}t\mathrm{d}u$$
$$\lesssim \int_{1/2}^{1}\int_{\partial B} |\mathfrak{B}_1 s(t,\hat{x})|^2 \,\mathrm{d}\hat{x}\mathrm{d}t.$$

The second summand can be controlled as in the usual proof of the Neumann–Poincare inequality: We have

$$\begin{aligned}|\pi s(t,\cdot) - \pi s(u,\cdot)| &= \left|\int_0^1 \partial_v \pi s(t + v(t-u),\cdot)\,\mathrm{d}v\right| \\ &\leqslant \left|\int_0^1 \pi(\partial_t s)(t+v(t-u),\cdot)\,\mathrm{d}v\right| \\ &\lesssim \left(\int_0^1\int_{\partial B} |(\partial_t s)(t+v(t-u),\hat{x})|^2\,\mathrm{d}\hat{x}\mathrm{d}v\right)^{1/2}.\end{aligned}$$

Plugging this into the second summand and symmetry considerations yield

$$\begin{aligned}&\int_{1/2}^{1}\int_{1/2}^{1}\int_{\partial B} |\pi s(t,\cdot) - \pi s(u,\cdot)|^2\,\mathrm{d}\hat{x}\mathrm{d}u\mathrm{d}t \\ &\quad\lesssim \int_{1/2}^{1}\int_{1/2}^{1}\int_0^1\int_{\partial B} |(\partial_t s)(t+v(t-u),\hat{x})|^2\,\mathrm{d}\hat{x}\mathrm{d}v\mathrm{d}u\mathrm{d}t \\ &\quad\lesssim \int_{1/2}^{1}\int_{\partial B} |\partial_t s(t,\hat{x})|^2\,\mathrm{d}\hat{x}\mathrm{d}t.\end{aligned}$$

This finishes the proof. ■

11.2 A differential inequality

The following differential inequality for

$$\hat{s}_r := \log\bigl(e^{-\Pi_r s}e^{s}\bigr).$$

lies at the heart of the proof of Proposition 11.2. By construction, the section $\hat{s}_r$ is self-adjoint with respect to He^s as well as $He^{\Pi_r s}$, and

$$He^s = \bigl(He^{\Pi_r s}\bigr)e^{\hat{s}_r}.$$

Proposition 11.6. *The section $\hat{s}_r$ satisfies*

$$|\mathfrak{B}s| \lesssim |\mathfrak{B}\hat{s}_r|, \quad |\hat{s}_r| \lesssim |s - \Pi_r s|, \quad \textit{and} \quad |\mathfrak{B}\hat{s}_r|^2 \lesssim r^{-2+\beta} - \Delta|\hat{s}_r|^2$$

for some $\beta > 0$.

The proof relies on the following identity.

Proposition 11.7. *We have*

$$\langle \mathfrak{m}(He^s) - \mathfrak{m}(H), s\rangle = \frac{1}{4}\Delta|s|^2 + \frac{1}{2}|v(-s)\nabla_{A_H}s|^2 + \frac{1}{2}|v(-s)[\phi_H, s]|^2 + \frac{1}{2}|v(-s)[\xi_H, s]|^2$$

with

$$v(s) = \sqrt{\frac{e^{\mathrm{ad}_s} - \mathrm{id}}{\mathrm{ad}_s}} \in \mathrm{End}(\mathfrak{gl}(E)).$$

Proof. We prove the analogous formula in dimension four. We have

$$\partial_{A_{He^s}} = e^{-s}\partial_{A_H}e^s = \partial_H + \Upsilon(-s)\partial_H s \quad \text{and} \quad \varphi^{*,He^s} = e^{-s}\varphi^{*,H}e^s$$

with

$$\Upsilon(s) = \frac{e^{\mathrm{ad}_s} - \mathrm{id}}{\mathrm{ad}_s}.$$

Set

$$D \coloneqq \bar{\partial} + i\varphi \quad \text{and} \quad \bar{D}_H \coloneqq \partial_H - i\varphi^{*,H}.$$

The above formula asserts that

$$\bar{D}_{He^s} = e^{-s}\bar{D}_H e^s = \bar{D}_H + \Upsilon(-s)\bar{D}_H s.$$

Since

$$D + \bar{D}_H = \nabla_{A_H} + i\phi_H,$$

we have

$$\mathfrak{m}(H) = \frac{1}{2}i\Lambda[D, \bar{D}_H].$$

Therefore,

$$\begin{aligned}
\langle \mathfrak{m}(He^s) - \mathfrak{m}(H), s\rangle &= i\Lambda\langle D(\Upsilon(-s)\bar{D}_H s), s\rangle \\
&= i\Lambda\bar{\partial}\langle \Upsilon(-s)\bar{D}_H s, s\rangle + i\Lambda\langle \Upsilon(-s)\bar{D}_H s \wedge \bar{D}_H s\rangle \\
&= \partial^*\langle \bar{D}_H s, \Upsilon(s)s\rangle + |v(-s)\bar{D}_H s|^2 \\
&= \frac{1}{2}\partial^*\partial|s|^2 + |v(-s)\bar{D}_H s|^2 \\
&= \frac{1}{4}\Delta|s|^2 + \frac{1}{2}|v(-s)(\nabla_H + i[\phi, \cdot])s|^2.
\end{aligned}$$

■

Proof of Proposition 11.6. The first two estimates are elementary. To prove the last estimate we argue as follows. Set

$$a \coloneqq \nabla_{A_H} - \nabla_{A_\mathrm{k}} \quad \text{and} \quad \hat{\xi} \coloneqq \xi_H - \xi_\mathrm{k}.$$

By (11.3) and since $\Pi_r s$ lies in the kernel of $\mathfrak{B}$, for some $\beta > 0$

$$\begin{aligned}
|\mathfrak{B}\hat{s}_r|^2 &\lesssim |\nabla_{A_H}\hat{s}_r|^2 + |[\xi_H, \hat{s}_r]|^2 + r^{-2+2\beta} \\
&\lesssim |\nabla_{A_{He^{\Pi_r s}}}\hat{s}_r|^2 + |[\xi_{He^{\Pi_r s}}, \hat{s}_r]|^2 + r^{-2+2\beta}.
\end{aligned}$$

Therefore, it suffices to estimate $|\nabla_{A_{He^{\Pi_r s}}}\hat{s}_r|^2 + |[\xi_{He^{\Pi_r s}}, \hat{s}_r]|^2$.

Since $\hat{s}_r$ is bounded, $v(\hat{s}_r)$ is bounded away from zero. Hence, by Proposition 11.7 with $He^{\Pi_r s}$ instead of H and $\hat{s}_r$ instead of s,

$$|\nabla_{A_{He^{\Pi_r s}}}\hat{s}_r|^2 + |[\phi_{He^{\Pi_r}}, \hat{s}_r]|^2 \lesssim |\mathfrak{m}(He^s)| + |\mathfrak{m}(He^{\Pi_r s})| - \Delta|\hat{s}_r|^2.$$

It follows from (11.3) that $|\mathfrak{m}(H)| = O(r^{-2+\beta})$. Moreover, since $\Pi_r s$ lies in the kernel of $\mathfrak{B}$, $|\mathfrak{m}(He^{\Pi_r s})| = O(r^{-2+\beta})$. Furthermore, $\mathfrak{m}(He^s) = 0$, Putting all of the above together yields the asserted estimate. ■

11.3 Proof of Proposition 11.2

Set

$$g(r) := \int_{B_r} |x|^{-1}|\mathfrak{B}s|^2$$

with $|x|$ denoting the distance to the center of the ball B_r. The upcoming three steps show that $g(r) \lesssim r^{2\alpha}$ for some $\alpha > 0$. This implies the assertion.

Step 1. *The function $|x|^{-1}|\mathfrak{B}s|^2$ is integrable; in particular: $g \leqslant c$.*

Fix a smooth function $\chi\colon [0,\infty) \to [0,1]$ which is equal to one on $[0,1]$ and vanishes outside $[0,2]$. Set $\chi_r(\cdot) := \chi(|\cdot|/r)$. Denote by G the Green's function of B centered at 0. For $r > \varepsilon > 0$, using Proposition 11.6, we have

$$\begin{aligned}\int_{B_r\setminus B_\varepsilon} |x|^{-1}|\mathfrak{B}s|^2 &\lesssim \int_{B_{2r}\setminus B_{\varepsilon/2}} \chi_r(1-\chi_{\varepsilon/2})G(r^{-2+\beta} - \Delta|\hat{s}_r|^2) \\ &\lesssim r^\beta + r^{-3}\int_{B_{2r}\setminus B_r} |\hat{s}_r|^2 + \varepsilon^{-3}\int_{B_\varepsilon\setminus B_{\varepsilon/2}} |\hat{s}_r|^2.\end{aligned}$$

Since s is bounded, the right-hand side is bounded independent of ε. This proves the integrability of $|x|^{-1}|\mathfrak{B}s|^2$ and the yields a bound on g.

Step 2. *There are constants $\gamma \in [0,1)$ and $c > 0$ such that*

$$g(r) \leqslant \gamma g(2r) + cr^\beta.$$

Continue the inequality from the previous step using the Neumann–Poicaré estimate (11.5) as

$$\begin{aligned}\int_{B_r\setminus B_\varepsilon} |x|^{-1}|\mathfrak{B}s|^2 &\lesssim r^\beta + r^{-3}\int_{B_{2r}\setminus B_r} |s - \Pi_r s|^2 + \varepsilon^{-3}\int_{B_\varepsilon\setminus B_{\varepsilon/2}} |s-\Pi_r s|^2 \\ &\lesssim r^\beta + r^{-1}\int_{B_{2r}\setminus B_r} |\mathfrak{B}s|^2 + \varepsilon^{-1}\int_{B_\varepsilon\setminus B_{\varepsilon/2}} |\mathfrak{B}s|^2 \\ &\lesssim r^\beta + g(2r) - g(r) + g(\varepsilon).\end{aligned}$$

By Lebesgue's monotone convergence theorem, the last term vanishes as ε tends to zero. Therefore,

$$g(r) \lesssim g(2r) - g(r) + r^\beta$$

Step 3. *For some $\alpha > 0$, $g \lesssim r^{2\alpha}$.*

This follows from the preceding steps by an elementary argument; see, e.g., [JW19, Step 3 in the proof of Proposition C.2]. ■

A Sequences of Hecke modifications

This appendix discusses the extension of Theorem 5.10 to sequences of Hecke modifications. Let Σ be a closed Riemann surface, let $(\mathscr{E}_0, \varphi_0)$ be a Higgs bundle over Σ of rank r, let $z_1, \ldots, z_n \in \Sigma$, and let $\mathbf{k}_1, \ldots, \mathbf{k}_n \in \mathbf{Z}^r$ satisfying (2.6).

Definition A.1. A **sequence of Hecke modifications** of $(\mathscr{E}_0, \varphi_0)$ at $z_1, \ldots, z_n$ of type $\mathbf{k}_1, \ldots, \mathbf{k}_n$ consists of a Hecke modification

$$\eta_i \colon (\mathscr{E}_{i-1}, \varphi_{i-1})|_{\Sigma\setminus\{z_i\}} \cong (\mathscr{E}_i, \varphi_i)|_{\Sigma\setminus\{z_i\}}$$

at z_i of type $\mathbf{k}_i$ for every $i = 1, \ldots, n$. An **isomorphism** between two sequences of Hecke modification $(\mathscr{E}_i, \varphi_i; \eta_i)_{i=1}^n$ and $(\tilde{\mathscr{E}}_i, \tilde{\varphi}_i; \tilde{\eta}_i)_{i=1}^n$ consists of an isomorphism

$$\zeta_i \colon (\mathscr{E}_i, \varphi_i) \to (\tilde{\mathscr{E}}_i, \tilde{\varphi}_i)$$

of Higgs bundles such that

$$\zeta_{i-1}\eta_i = \tilde{\eta}_i\zeta_i$$

for every $i = 1, \ldots, n$ and with $\zeta_0 \coloneqq \mathrm{id}_{\mathscr{E}_0}$. We denote by

$$\mathscr{M}^{\mathrm{Hecke}}(\mathscr{E}_0, \varphi_0; z_1, \ldots, z_n, \mathbf{k}_1, \ldots, \mathbf{k}_n)$$

the set of all isomorphism classes of sequences of Hecke modifications of $(\mathscr{E}_0, \varphi_0)$ at $z_1, \ldots, z_n$ of type $\mathbf{k}_1, \ldots, \mathbf{k}_n$. •

Denote by E_0 the complex vector bundle underlying $\mathscr{E}_0$. Henceforth, we assume that H_0 is a Hermitian metric on E_0. Furthermore, fix

$$0 < y_1 < y_2 < \cdots < y_n < 1.$$

As in Proposition 4.1, there exists a Hermitian vector bundle (E, H) over

$$M \coloneqq [0,1] \times \Sigma\setminus\{(y_1, z_1), \ldots, (y_n, z_n)\}$$

together with a framing Ψ_i at (y_i, z_i) of type $\mathbf{k}_i$ for every $i = 1, \ldots, n$. Any two choices of $(E, H; \Psi_1, \ldots, \Psi_n)$ are isomorphic. Throughout the remainder of this appendix, we fix one such choice.

Definition A.2. Denote by $\mathscr{C}^{\mathrm{EBE}}(A_0, \phi_0; y_1, z_1, \ldots, y_n, z_n, \mathbf{k}_1, \ldots, \mathbf{k}_n)$ the set of triples

$$A \in \mathscr{A}(E, H), \quad \phi \in \Omega^1(M, \mathfrak{u}(E, H)), \quad \text{and} \quad \xi \in \Omega^0(M, \mathfrak{u}(E, H))$$

satisfying the extended Bogomolny equation (3.2), as well as

$$i(\partial_y)\phi = 0,$$

and the boundary conditions

$$A|_{\{0\}\times\Sigma} = A_0, \quad \phi|_{\{0\}\times\Sigma} = \phi_0, \quad \text{and} \quad \xi|_{\{1\}\times\Sigma} = 0.$$

Denote by

$$\mathscr{G} \subset \mathscr{G}(E,H)$$

the subgroup of unitary gauge transformations of (E,H) which are singularity preserving at $(y_1,z_1),\dots,(y_n,z_n)$ and restrict to the identity on $\{0\}\times\Sigma$. Set

$$\mathscr{M}^{\mathrm{EBE}}(A_0,\phi_0;y_1,z_1,\dots,y_n,z_n,\mathbf{k}_1,\dots,\mathbf{k}_n) \coloneqq \mathscr{C}^{\mathrm{EBE}}(A_0,\phi_0;y_1,z_1,\dots,y_n,z_n,\mathbf{k}_1,\dots,\mathbf{k}_n)/\mathscr{G}. \quad \bullet$$

Let $(A,\phi,\xi) \in \mathscr{C}^{\mathrm{EBE}}(A_0,\phi_0;y_1,z_1,\dots,y_n,z_n,\mathbf{k}_1,\dots,\mathbf{k}_n)$. Let

$$y_1 < m_1 < y_2 < m_2 < \dots < y_n < m_n \coloneqq 1.$$

The scattering map construction from Section 5 restricted to $[0,m_1]\times\Sigma$ yields a Hecke modification $(\mathscr{E}_1,\varphi_1;\eta_1)$ of $(\mathscr{E}_0,\varphi_0)$ at z_1 of type $\mathbf{k}_1$. Similarly, we obtain a Hecke modification $(\mathscr{E}_i,\varphi_i;\eta_1)$ of $(\mathscr{E}_{i-1},\varphi_{i-1})$ at z_i of type $\mathbf{k}_i$ for every $i=1,\dots,n$. A different choice of $\tilde{m}_i \in (y_i,y_{i+1})$ may yield a different Hecke modification $(\tilde{\mathscr{E}}_i,\tilde{\varphi}_i;\tilde{\eta}_i)$. However, these Hecke modifications are isomorphic via the scattering map from m_i to $\tilde{m}_i$. Therefore, we obtain a map

$$\mathscr{C}^{\mathrm{EBE}}(A_0,\phi_0;y_1,z_1,\dots,y_n,z_n,\mathbf{k}_1,\dots,\mathbf{k}_n) \to \mathscr{M}^{\mathrm{Hecke}}(\mathscr{E}_0,\varphi_0;z_1,\dots,z_n,\mathbf{k}_1,\dots,\mathbf{k}_n).$$

This map is $\mathscr{G}$–invariant. We have the following extension of Theorem 5.10.

Theorem A.3. *The map*

$$\mathscr{M}^{\mathrm{EBE}}(A_0,\phi_0;y_1,z_1,\dots,y_n,z_n,\mathbf{k}_1,\dots,\mathbf{k}_n) \to \mathscr{M}^{\mathrm{Hecke}}(\mathscr{E}_0,\varphi_0;z_1,\dots,z_n,\mathbf{k}_1,\dots,\mathbf{k}_n)$$

induced by the scattering map construction is a bijection.

Proof. The proof is essentially the same as that of Theorem 5.10. The notion of parametrized Hecke modifications can be extended to parametrized sequences of Hecke modifications yielding a moduli space $\mathscr{M}^{\widetilde{\mathrm{Hecke}}}(\mathscr{E}_0,\varphi_0;y_1,z_1,\dots,y_n,z_n,\mathbf{k}_1,\dots,\mathbf{k}_n)$. As in the proof of Proposition 6.2, one shows that the scattering map yields a bijection

$$\mathscr{M}^{\widetilde{\mathrm{Hecke}}}(\mathscr{E}_0,\varphi_0;y_1,z_1,\dots,y_n,z_n,\mathbf{k}_1,\dots,\mathbf{k}_n) \to \mathscr{M}^{\mathrm{Hecke}}(\mathscr{E}_0,\varphi_0;z_1,\dots,z_n,\mathbf{k}_1,\dots,\mathbf{k}_n).$$

Finally, the arguments from Section 7, Section 8, Section 9, Section 10, and Section 11 show that the obvious map

$$\mathscr{M}^{\mathrm{EBE}}(A_0,\phi_0;y_1,z_1,\dots,y_n,z_n,\mathbf{k}_1,\dots,\mathbf{k}_n) \to \mathscr{M}^{\widetilde{\mathrm{Hecke}}}(\mathscr{E}_0,\varphi_0;y_1,z_1,\dots,y_n,z_n,\mathbf{k}_1,\dots,\mathbf{k}_n)$$

is a bijection. ■

Remark A.4. If $\varphi = 0$, then the above reduces to the notion of a sequence of Hecke modifications of a holomorphic vector bundle; see, e.g., [Won13, Section 1.5.1; Boo18, Section 2.4]. ♣

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