Equivariant Brill–Noether theory for elliptic operators and super-rigidity of J–holomorphic maps

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Abstract

The space of Fredholm operators of fixed index is stratified by submanifolds according to the dimension of the kernel. Geometric considerations often lead to questions about the intersections of concrete families of elliptic operators with these submanifolds: are the intersections non-empty? are they smooth? what are their codimensions? The purpose of this article is to develop tools to address these questions in equivariant situations. An important motivation for this work are transversality questions for multiple covers of J–holomorphic maps. As an application, we use our framework to give a concise exposition of Wendl's proof of the super-rigidity conjecture.

Introduction

Let X and Y be two finite dimensional vector spaces. The space $\mathrm{Hom}(X, Y)$ is stratified by the submanifolds

$$\mathscr{H}_r \coloneqq \{L \in \mathrm{Hom}(X, Y) : \mathrm{rk}\, L = r\}$$

of codimension

$$\operatorname{codim} \mathscr{H}_r = (\dim X - r)(\dim Y - r).$$

This generalizes to infinite dimensions as follows. Let X and Y be two Banach spaces. The space of Fredholm operators from X to Y, denoted by $\mathscr{F}(X, Y)$, is stratified by the submanifolds

$$\mathscr{F}_{d,e} \coloneqq \{L \in \mathscr{F}(X, Y) : \dim \ker L = d \text{ and } \dim \operatorname{coker} L = e\}$$

of codimension

$$\operatorname{codim} \mathscr{F}_{d,e} = de.$$

In many geometric problems, especially in the study of moduli spaces in algebraic geometry, gauge theory, and symplectic topology, one is led to consider families of Fredholm operators $D\colon \mathscr{P} \to \mathscr{F}(X, Y)$ parametrized by a Banach manifold $\mathscr{P}$, and to analyze the subsets $D^{-1}(\mathscr{F}_{d,e})$.

The archetypal example is Brill–Noether theory in algebraic geometry. Let Σ be a closed, connected Riemann surface of genus g. Denote by $\mathrm{Pic}(\Sigma)$ the Picard group of isomorphism classes of holomorphic line bundles $\mathscr{L} \to \Sigma$. Brill–Noether theory is concerned with the study of the subsets $G^r_d \subset \mathrm{Pic}(\Sigma)$, called the Brill–Noether loci, defined by

$$G^r_d \coloneqq \left\{ [\mathscr{L}] \in \mathrm{Pic}(\Sigma) : \deg(\mathscr{L}) = d \text{ and } \dim_{\mathbf{C}} H^0(\Sigma, \mathscr{L}) = r+1 \right\}.$$

The fundamental results of this theory deal with the questions of whether G^r_d is non-empty, smooth, and of the expected complex codimension.

This connects to the previous discussion as follows. Let L be a Hermitian line bundle of degree d over Σ. Denote by $\mathscr{A}(L)$ the space of unitary connections on L. The complex gauge group $\mathscr{G}^{\mathbf{C}}(L)$ acts on $\mathscr{A}(L)$ and the quotient $\mathscr{A}(L)/\mathscr{G}^{\mathbf{C}}(L)$ is biholomorphic to $\mathrm{Pic}^d(\Sigma)$, the component of $\mathrm{Pic}(\Sigma)$ parametrizing holomorphic line bundles of degree d. Define the family of Fredholm operators

$$\bar\partial \colon \mathscr{A}(L) \to \mathscr{F}(\Gamma(L), \Omega^{0,1}(\Sigma, L))$$

by assigning to every connection A the Dolbeault operator $\bar\partial_A = \nabla^{0,1}_A$. Set

$$\tilde G^r_d \coloneqq \bar\partial^{-1}(\mathscr{F}_{r+1,g-d+r}).$$

It follows from the Riemann–Roch Theorem and Hodge theory that the Brill–Noether loci can be described as the quotients

$$G^r_d = \tilde G^r_d / \mathscr{G}^{\mathbf{C}}(L).$$

If G^r_d is non-empty, then

$$\operatorname{codim}_{\mathbf{C}} G^r_d = \operatorname{codim}_{\mathbf{C}} \tilde G^r_d \leqslant (r+1)(g-d+r).$$

This is an immediate consequence of the definition of $\tilde G^r_d$ and $\operatorname{codim}_{\mathbf{C}} \mathscr{F}_{d,e} = de$. Ideally, every G^r_d is smooth of complex codimension $(r+1)(g-d+r)$. This is not always true, but Gieseker [Gie82] proved that it holds for generic Σ; see also [EH83; Laz86]. Furthermore, Kempf [Kem71] and Kleiman and Laksov [KL72; KL74] proved that if $(r+1)(g-d+r) \leqslant g$, then G^r_d is non-empty. For an extensive discussion of Brill–Noether theory in algebraic geometry we refer the reader to [ACGH85].

By analogy, for a family of Fredholm operators $D \colon \mathscr{P} \to \mathscr{F}(X, Y)$ one might ask:

(1) When are the subsets $D^{-1}(\mathscr{F}_{d,e})$ non-empty?

(2) When are they smooth submanifolds of $\mathscr{P}$?

(3) What are their codimensions?

Index theory and theory of spectral flow sometimes give partial results regarding (1). A simple answer to (2) and (3) is that $D^{-1}(\mathscr{F}_{d,e})$ is smooth and of codimension de if the map D is transverse to $\mathscr{F}_{d,e}$. However, for many naturally occurring families of elliptic operators this condition does not hold. For example, if D is a family of elliptic operators over a manifold M and $\underline{V}$ is a local system, then the family $D^{\underline{V}}$ of the elliptic operators D twisted by $\underline{V}$ often is not transverse to $\mathscr{F}_{d,e}$ even if D is. Related issues arise for families of elliptic operators pulled back by a covering map $\pi\colon \tilde{M} \to M$. The purpose of this article is to give useful tools for answering (2) and (3) which apply to these equivariant situations. This theory is developed in Part 1.

The issues discussed above are well-known to arise from multiple covers in the theory of J–holomorphic maps in symplectic topology. In fact, our motivation for writing this article came from trying to understand Wendl's proof of Bryan and Pandharipande's super-rigidity conjecture for J–holomorphic maps [Wen19b]. The theory developed in Part 1 is essentially an abstraction of Wendl's ideas, some of which can themselves be traced back to Taubes [Tau96] and Eftekhary [Eft16]. In Part 2 we use this theory to give a concise exposition of the proof of the super-rigidity conjecture. The main results of Part 2 are contained in [Wen19b] and most of the proofs closely follow Wendl's approach. There are, however, two key differences:

(1) Our discussion consistently uses the language of local systems. This appears to us to be more natural for the problem at hand. It also avoids the use of representation theory and covering theory. In particular, there is no need to take special care of non-normal covering maps.

(2) Our approach to dealing with branched covering maps is geometric: branched covering map between Riemann surfaces are reinterpreted as unbranched covering maps between orbifold Riemann surfaces. This is to be compared with Wendl's analytic approach which uses suitable weighted Sobolev spaces on punctured Riemann surfaces. One feature of our approach is that it leads to a simple proof of the crucial index theorem; cf. Section 2.B and [Wen19b, Theorem 4.1].

We expect the theory developed in Part 1 to have many applications outside of the theory of J–holomorphic maps. In future work we plan to study transversality for multiple covers of calibrated submanifolds in manifolds with special holonomy, such as associative submanifolds in G_2–manifolds.

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Contents

Part 1

Equivariant Brill–Noether theory

Throughout this part, let (M, g) be a closed, connected, oriented Riemannian orbifold of dimension $\dim M = n$, and let E and F be Euclidean vector bundles of rank $\operatorname{rk} E = \operatorname{rk} F = r$ over M equipped with orthogonal covariant derivatives.[1] Here and throughout this article dim and rk denote the dimension and rank over the real numbers. If dimension and rank are to be taken over a different field, then this is indicated by a subscript. $\Gamma(E)$ denotes the space of smooth sections of E. For an open subset $U \subset M$, $\Gamma(U, E)$ and $\Gamma_c(U, E) \subset \Gamma(E)$ denote the spaces of smooth sections of E defined over U and with support in U respectively. For $k \in \mathbf{N}_0 = \{0, 1, 2, \ldots\}$, denote by $W^{k,2}\Gamma(E)$ the Sobolev completion of $\Gamma(E)$ with respect to the $W^{k,2}$ norm induced by the Euclidean metric and covariant derivative. For $k \in \mathbf{N}$ set $W^{-k,2}\Gamma(E) \coloneqq W^{k,2}\Gamma(E)^*$. Set $L^2\Gamma(E) \coloneqq W^{0,2}\Gamma(E)$. Denote by $L^1\Gamma(E)$ the completion of $\Gamma(E)$ with respect to the L^1 norm. (Analogous notation is used, instead of E, for F etc.)

1.1 Brill–Noether loci

Let us begin by discussing the non-equivariant theory.

Definition 1.1.1. Let $k \in \mathbf{N}_0$. A **family of linear elliptic differential operators of order** k consists of a Banach manifold $\mathscr{P}$ and a smooth map

$$D\colon \mathscr{P} \to \mathscr{F}(W^{k,2}\Gamma(E), L^2\Gamma(F))$$

such that for every $p \in \mathscr{P}$ the operator $D_p \coloneqq D(p)$ is a linear elliptic differential operator.[2,3] •

Definition 1.1.2. Let $(D_p)_{p\in\mathscr{P}}$ be a family of linear elliptic differential operators. For $d, e \in \mathbf{N}_0$ define the **Brill–Noether locus** $\mathscr{P}_{d,e}$ by

$$\mathscr{P}_{d,e} \coloneqq \left\{p \in \mathscr{P} : \dim\ker D_p = d \text{ and } \dim\operatorname{coker} D_p = e\right\}. \qquad \bullet$$

Remark 1.1.3. Let $(D_p)_{p\in\mathscr{P}}$ be a family of linear elliptic operators of index $i \in \mathbf{Z}$. If $\mathscr{P}_{d,e} \neq \varnothing$, then $d - e = i$; in particular: $d \geqslant i$ and $e \geqslant -i$. ♣

[1] Remark 1.2.2 explains why we allow orbifolds. For the purposes of this article, the category of orbifolds is the one constructed by Moerdijk [Moe02] via groupoids; see also [ALR07]. [LU04, Section 5] compares Moerdijk's approach with the original approach via orbifold charts developed by Satake [Sat56] and Thurston [Thu02]. [SY19, Section 3] discusses differential operators and Sobolev spaces on orbifolds.

[2] Banach manifolds are assumed to be Hausdorff, paracompact, and separable. This is required in Section 1.B where the Sard–Smale Theorem is used.

[3] Denote by $\nabla^\ell\colon \Gamma(E) \to \Gamma(T^*M^\ell \otimes E)$ the ℓ–th covariant derivative. If $D_p \coloneqq \sum_{\ell=0}^k a_\ell(p,\cdot)\nabla^\ell$ with $a_\ell(p,\cdot)$ a section of $\operatorname{Hom}(T^*M^{\otimes\ell} \otimes E, F))$, then it suffices to prove that a_ℓ defines smooth section of $\operatorname{pr}_2^*\operatorname{Hom}(T^*M^{\otimes\ell} \otimes E, F)$ over $\mathscr{P} \times M$ to establish that D is smooth. Indeed, Section 1.6 requires this stronger hypothesis; cf. Definition 1.6.1.

The following elementary fact from the theory of Fredholm operators reduces the discussion to the finite-dimensional case. As in the introduction, if X and Y are Banach spaces, then $\mathscr{L}(X, Y)$ denotes the Banach space of bounded linear maps from X to Y equipped with the operator norm, and $\mathscr{F}(X, Y) \subset \mathscr{L}(X, Y)$ denotes the open subset of Fredholm operators from X to Y.

Lemma 1.1.4 (cf. Koschorke [Kos68, Chapter I §1.b]). *Let X and Y be Banach spaces. For every $L \in \mathscr{F}(X, Y)$ there is an open neighborhood $\mathscr{U} \subset \mathscr{F}(X, Y)$ and a smooth map $\mathcal{S}\colon \mathscr{U} \to$ $\operatorname{Hom}(\ker L, \operatorname{coker} L)$ such that for every $T \in \mathscr{U}$ there are isomorphisms*

$$\ker T \cong \ker \mathcal{S}(T) \quad \textit{and} \quad \operatorname{coker} T \cong \operatorname{coker} \mathcal{S}(T);$$

furthermore, the derivative of S at L,

$$\mathrm{d}_L\mathcal{S}\colon T_L\mathscr{F}(X, Y) \to \operatorname{Hom}(\ker L, \operatorname{coker} L)$$

satisfies

$$\mathrm{d}_L\mathcal{S}(\hat{L})s = \hat{L}s \mod \operatorname{im} L$$

for every $\hat{L} \in T_L\mathscr{F}(X, Y) = \mathscr{L}(X, Y)$.

Proof. Pick a complement $\operatorname{coim} L$ of $\ker L$ in X and a lift of $\operatorname{coker} L$ to Y. With respect to the splittings $X = \operatorname{coim} L \oplus \ker L$ and $Y = \operatorname{im} L \oplus \operatorname{coker} L$ every $T \in \mathscr{F}(X, Y)$ can be written as

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}.$$

By construction, L_{11} is invertible, and the remaining components of L vanish.

Choose an open neighborhood $\mathscr{U}$ of L in $\mathscr{F}(X, Y)$ such that for every $T \in \mathscr{U}$ the operator T_{11} is invertible. Define $\mathcal{S}\colon \mathscr{U} \to \operatorname{Hom}(\ker L, \operatorname{coker} L)$ by

$$\mathcal{S}(T) \coloneqq T_{22} - T_{21}T_{11}^{-1}T_{12}.$$

A brief computation shows that for every $T \in \mathscr{U}$

$$\Phi T \Psi = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathcal{S}(T) \end{pmatrix} \quad \text{with} \quad \Phi \coloneqq \begin{pmatrix} T_{11}^{-1} & 0 \\ -T_{21}T_{11}^{-1} & \mathbf{1} \end{pmatrix} \quad \text{and} \quad \Psi \coloneqq \begin{pmatrix} \mathbf{1} & -T_{11}^{-1}T_{12} \\ 0 & \mathbf{1} \end{pmatrix};$$

hence, $\ker T \cong \ker \mathcal{S}(T)$ and $\operatorname{coker} T \cong \operatorname{coker} \mathcal{S}(T)$.

The formula for $\mathrm{d}_L S$ is evident from the fact that L_{21} and L_{12} vanish. ∎

Lemma 1.1.4 together with the Regular Value Theorem immediately imply the following.

Theorem 1.1.5. *Let $(D_p)_{p\in\mathscr{P}}$ be a family of linear elliptic differential operators. For $p \in \mathscr{P}$ define $\Lambda_p\colon T_p\mathscr{P} \to \operatorname{Hom}(\ker D_p, \operatorname{coker} D_p)$ by*

$$\Lambda_p(\hat{p})s \coloneqq \mathrm{d}_pD(\hat{p})s \mod \operatorname{im} D_p.$$

Let $d, e \in \mathbf{N}_0$ and $p \in \mathscr{P}_{d,e}$. If Λ_p is surjective, then following hold:

(1) *There is an open neighborhood $\mathscr{U}$ of $p \in \mathscr{P}$ such that $\mathscr{P}_{d,e} \cap \mathscr{U}$ is a submanifold of codimension*

$$\operatorname{codim}(\mathscr{P}_{d,e} \cap \mathscr{U}) = de.$$

(2) *$\mathscr{P}_{\tilde{d},\tilde{e}} \neq \varnothing$ for every $\tilde{d}, \tilde{e} \in \mathbf{N}_0$ with $\tilde{d} \leqslant d, \tilde{e} \leqslant e$, and $\tilde{d} - \tilde{e} = d - e$.* ■

Remark 1.1.6. If E and F are Hermitian vector bundles and $(D_p)_{p\in\mathscr{P}}$ is a family of complex linear elliptic differential operators, then the map Λ_p factors through $\operatorname{Hom}_{\mathbf{C}}(\ker D_p, \operatorname{coker} D_p)$. Therefore, the hypothesis of Theorem 1.1.5 cannot be satisfied (unless it holds trivially). Of course, this issue is rectified by replacing $\mathbf{R}$ with $\mathbf{C}$ throughout the above discussion. ♣

Example 1.1.7 (Brill–Noether theory for holomorphic line bundles over a Riemann surface). Let Σ be a closed, connected Riemann surface of genus g. Let L be a Hermitian line bundle of degree d over Σ. Denote by $\mathscr{A}(L)$ the space of unitary connections on L.[4] Define the family of complex linear elliptic differential operators

$$\bar{\partial}\colon \mathscr{A}(L) \to \mathscr{F}(W^{1,2}\Gamma(L), L^2\Omega^{0,1}(\Sigma, L))$$

by assigning to every connection A the Dolbeault operator $\bar{\partial}_A := \nabla_A^{0,1}$.

Let $A \in \mathscr{A}(L)$. The map $\Lambda_A\colon T_A\mathscr{A}(L) \to \operatorname{Hom}_{\mathbf{C}}(\ker \bar{\partial}_A, \operatorname{coker} \bar{\partial}_A)$ can be described concretely as follows. Since the derivative of the map $\bar{\partial}$ is $\mathrm{d}_A\bar{\partial}(a) = a^{0,1}$, the map Λ_A factors through the isomorphism $T_A\mathscr{A}(L) = \Omega^1(\Sigma, i\mathbf{R}) \cong \Omega^{0,1}(\Sigma)$ defined by $a \mapsto a^{0,1}$. Denote by $\mathscr{L}$ the holomorphic line bundle associated with $\bar{\partial}_A$. By Serre duality,

$$\operatorname{coker} \bar{\partial}_A = H^1(\Sigma, \mathscr{L}) \cong H^0(\Sigma, K_\Sigma \otimes_{\mathbf{C}} \mathscr{L}^*)^*;$$

hence:

$$\operatorname{Hom}_{\mathbf{C}}(\ker \bar{\partial}_A, \operatorname{coker} \bar{\partial}_A) \cong H^0(\Sigma, \mathscr{L})^* \otimes_{\mathbf{C}} H^0(\Sigma, K_\Sigma \otimes_{\mathbf{C}} \mathscr{L}^*)^*.$$

The Petri map

$$\varpi_{\mathscr{L}}\colon H^0(\Sigma, \mathscr{L}) \otimes_{\mathbf{C}} H^0(\Sigma, K_\Sigma \otimes \mathscr{L}^*) \to H^0(\Sigma, K_\Sigma) \tag{1.1.8}$$

is induced by the isomorphism $\mathscr{L} \otimes_{\mathbf{C}} \mathscr{L}^* \cong \mathscr{O}_\Sigma$. The adjoint of Λ_A is the composition of the Petri map $\varpi_{\mathscr{L}}$ with the inclusion $H^0(\Sigma, K_\Sigma) \hookrightarrow \Omega^{1,0}(\Sigma)$. Here the duality between $\Omega^{1,0}(\Sigma)$ and $\Omega^{0,1}(\Sigma)$ is given by integration.

As a consequence, Λ_A is surjective if and only if $\varpi_{\mathscr{L}}$ is injective. If $\varpi_{\mathscr{L}}$ is injective for every $[\mathscr{L}] \in \operatorname{Pic}^d(\Sigma)$, then

$$\tilde{G}_d^r := \mathscr{A}(L)_{r+1,g-d+r}$$

is a complex submanifold of codimension $(r+1)(g-d+r)$; therefore, so is the Brill–Noether locus

$$G_d^r := \tilde{G}_d^r/\mathscr{G}^{\mathbf{C}}(L) \cong \left\{ \mathscr{L} \in \operatorname{Pic}^d(\Sigma) : \begin{array}{l} \dim H^0(\Sigma, \mathscr{L}) = r+1 \text{ and} \\ \dim H^1(\Sigma, \mathscr{L}) = g-d+r \end{array} \right\};$$

cf. [ACGH85, Lemma 1.6, Chapter IV]. ♠

[4] Strictly speaking, to be in the situation of Definition 1.1.1, $\mathscr{A}(L)$ should be replaced by a suitable Banach space completion.

This example motivates the following definitions, which are particularly appropriate for first order operators appearing in geometric applications.

Definition 1.1.9. Let $U \subset M$ be an open subset. A family of linear elliptic differential operators $(D_p)_{p\in\mathscr{P}}$ is **flexible in** U **at** $p_\star \in \mathscr{P}$ if for every $A \in \Gamma_c(U, \mathrm{Hom}(E, F))$ there is a $\hat{p} \in T_{p_\star}\mathscr{P}$ such that

$$\mathrm{d}_{p_\star}D(\hat{p})s = As \mod \operatorname{im} D_{p_\star}$$

for every $s \in \ker D_{p_\star}$. •

Definition 1.1.10. Let $D\colon W^{k,2}\Gamma(E) \to L^2\Gamma(F)$ be a linear differential operator. Set

$$E^\dagger \coloneqq E^* \otimes \Lambda^n T^*M \quad \text{and} \quad F^\dagger \coloneqq F^* \otimes \Lambda^n T^*M.$$

The **formal adjoint** of D is the linear differential operator $D^\dagger\colon L^2\Gamma(F^\dagger) \to W^{-k,2}\Gamma(E^\dagger)$ characterized by

$$\int_M \langle s, D^\dagger t\rangle = \int_M \langle Ds, t\rangle$$

for $s \in \Gamma(E)$ and $t \in \Gamma(F^\dagger)$. Here $\langle\cdot,\cdot\rangle$ denotes the canonical pairings $E \otimes E^\dagger \to \Lambda^n T^*M$ and $F \otimes F^\dagger \to \Lambda^n T^*M$. •

Definition 1.1.11. The **Petri map** $\varpi\colon \Gamma(E) \otimes \Gamma(F^\dagger) \to \Gamma(E \otimes F^\dagger)$ is defined by

$$\varpi(s \otimes t)(x) \coloneqq s(x) \otimes t(x).$$

Let $U \subset M$ be an open subset. A linear elliptic differential operator $D\colon W^{k,2}\Gamma(E) \to L^2\Gamma(F)$ satisfies **Petri's condition in** U if the map

$$\varpi_{D,U}\colon \ker D \otimes \ker D^\dagger \to L^1\Gamma(U, E \otimes F^\dagger)$$

induced by the Petri map is injective. •

Proposition 1.1.12. *Let $(D_p)_{p\in\mathscr{P}}$ be a family of linear elliptic differential operators. Let $U \subset M$ be an open subset. If $(D_p)_{p\in\mathscr{P}}$ is flexible in U at $p_\star \in \mathscr{P}$ and $D_{p_\star}$ satisfies Petri's condition in U, then the map $\Lambda_{p_\star}$ defined in Theorem 1.1.5 is surjective.*

Proof. Define the map $\mathrm{ev}_p\colon \Gamma_c(U, \mathrm{Hom}(E, F)) \to \mathrm{Hom}(\ker D_p, \operatorname{coker} D_p)$ by

$$\mathrm{ev}_p(A)s \coloneqq As \mod \operatorname{im} D_p.$$

$(D_p)_{p\in\mathscr{P}}$ is flexible in U at $p_\star \in \mathscr{P}$ if and only $\operatorname{im}\mathrm{ev}_{p_\star} \subset \operatorname{im}\Lambda_{p_\star}$. $D_{p_\star}$ satisfies Petri's condition in U if and only if $\mathrm{ev}_{p_\star}$ is surjective. To see this, observe the following. Since $\ker D^\dagger_{p_\star} \cong (\operatorname{coker} D_{p_\star})^*$, the pairing $\mathrm{Hom}(\ker D_{p_\star}, \operatorname{coker} D_{p_\star}) \otimes (\ker D_{p_\star} \otimes \ker D^\dagger_{p_\star}) \to \mathbf{R}$ induced by

$$\langle\!\langle s \otimes t, \tilde{A}\rangle\!\rangle \coloneqq \int_M t(\tilde{A}s)$$

is perfect, that is, it induces an isomorphism $\mathrm{Hom}(\ker D_{p_\star}, \operatorname{coker} D_{p_\star})^* \cong \ker D_{p_\star} \otimes \ker D_{p_\star}^\dagger$. There is a canonical perfect pairing $\langle\cdot,\cdot\rangle\colon\ \mathrm{Hom}(E,F) \otimes (E \otimes F^\dagger) \to \Lambda^n T^*M$. Evidently,

$$t(\mathrm{ev}_{p_\star}(A)s) = \langle A, \varpi(s \otimes t)\rangle.$$

Therefore, an element $B \in \ker D_{p_\star} \otimes \ker D_{p_\star}^\dagger$ annihilates $\operatorname{im}\mathrm{ev}_{p_\star}$ if and only if

$$\langle\!\langle \mathrm{ev}_{p_\star}(A), B\rangle\!\rangle = \int_M \langle A, \varpi(B)\rangle = 0$$

for every $A \in \Gamma_c(U, \mathrm{Hom}(E,F))$; that is: $\varpi(B) = 0$ in U. Therefore, $(\operatorname{im}\mathrm{ev}_{p_\star})^\perp = \ker \varpi_{D_{p_\star},U}$. ■

Remark 1.1.13. In Example 1.1.7, $\operatorname{im}\Lambda_p = \operatorname{im}\mathrm{ev}_p$ (with $U = \Sigma$, and $\mathbf{R}$ replaced with $\mathbf{C}$). Therefore, Λ_p being surjective is equivalent to Petri's condition. Furthermore, tracing through the isomorphisms identifies the restriction of Petri map ϖ to $\ker \bar\partial_A \otimes_{\mathbf{C}} \ker \bar\partial_A^*$ with the Petri map $\varpi_{\mathscr{L}}$. ♣

Flexibility is not a rare condition. Petri's condition appears to be more subtle. (By the uniqueness theorem for ordinary differential equations, it holds for first order linear elliptic differential operators on 1–manifolds. This is somewhat useful; see, e.g., [Eft19].) The upcoming Remark 1.1.14 (3) hints at the connection between Petri's condition and the unique continuation property. In Section 1.6, we revisit Petri's condition and discuss an algebraic criterion due to Wendl for Petri's condition to hold away from a subset of infinite codimension.

Remark 1.1.14. Assume the situation of Theorem 1.1.5. The following observations are useful in situations where the primary objective is to estimate the codimension of $\mathscr{P}_{d,e}$.

(1) Every $p \in \mathscr{P}_{d,e}$ has an open neighborhood $\mathscr{U}$ in $\mathscr{P}$ such that $\mathscr{P}_{d,e} \cap \mathscr{U}$ is contained in a submanifold of codimension $\operatorname{rk}\Lambda_p$.

(2) Let $\rho \in \mathbf{N}$ and let $U \subset M$ be an open subset. A linear elliptic differential operator $D\colon \Gamma(E) \to \Gamma(F)$ satisfies **Petri's condition up to rank ρ in U** if for every non-zero $B \in \ker D \otimes \ker D^\dagger$ of rank at most ρ the section $\varpi(B)$ does not vanish on U. (A **simple tensor** is non-zero tensor of the form $v \otimes w$. Every tensor B is a sum of simple tensors. The **rank** of B is the minimal number of simple tensors that sum to B.) If $D_{p_\star}$ satisfies this condition and $(D_p)_{p\in\mathscr{P}}$ is flexible in U at $p_\star \in \mathscr{P}$, then

$$\operatorname{rk}\Lambda_{p_\star} \geqslant \min\{\rho,d,e\}\max\{d,e\}.$$

Proof. Set $\sigma := \min\{\rho,d,e\}$. If $d \leqslant e$, then choose an injection $\mathbf{R}^\sigma \hookrightarrow \ker D_{p_\star}$ and set $H := \mathrm{Hom}(\mathbf{R}^\sigma, \operatorname{coker} D_{p_\star})$; otherwise, choose a surjection $\operatorname{coker} D_{p_\star} \twoheadrightarrow \mathbf{R}^\sigma$ and set $H := \mathrm{Hom}(\ker D_{p_\star}, \mathbf{R}^\sigma)$. In either case, composition defines a surjection

$$\pi_{p_\star}\colon\ \mathrm{Hom}(\ker D_{p_\star}, \operatorname{coker} D_{p_\star}) \to H.$$

The subspace $\operatorname{im}\pi_{p_\star}^* \subset \mathrm{Hom}(\ker D_{p_\star}, \operatorname{coker} D_{p_\star})^* \cong \ker D_{p_\star} \otimes \ker D_{p_\star}^\dagger$ consists of elements of rank at most $\sigma \leqslant \rho$. The argument of the proof of Proposition 1.1.12 thus shows that $\pi_{p_\star} \circ \Lambda_{p_\star}$ is surjective. Therefore, $\operatorname{rk}\Lambda_{p_\star} \geqslant \dim H = \min\{\rho,d,e\}\max\{d,e\}$. ■

(3) (This is due to Eftekhary [Eft16, Proof of Lemma 4.4].) Let $U \subset M$ be a non-empty open subset. Let $D\colon \Gamma(E) \to \Gamma(F)$ be a first linear elliptic differential operator of first order. If $\ker D$ and $\ker D^\dagger$ consists of continuous sections, and D and $D^\dagger$ have the weak unique continuation property, then D satisfies Petri's condition up to rank three in U.[5,6]

Proof. Every $B \in \ker D \otimes \ker D^\dagger$ can be written as $B = s_1 \otimes t_1 + \cdots + s_\rho \otimes t_\rho$ with $\rho \coloneqq \operatorname{rk} B$, and $s_1, \ldots, s_\rho \in \ker D$ and $t_1, \ldots, t_\rho \in \ker D^\dagger$ linearly independent. If $\rho = 1$ and $\varpi(B) = 0$, then s_1 or t_1 vanishes on an open subset; hence, by unique continuation, $s_1 = 0$ or $t_1 = 0$: a contradiction.

Henceforth, assume that $\rho \geqslant 2$. Define $\delta, \varepsilon\colon U \to \mathbf{N}_0$ by $\delta(x) \coloneqq \dim\langle s_1(x), \ldots, s_\rho(x)\rangle$ and $\varepsilon(x) \coloneqq \dim\langle t_1(x), \ldots, t_\rho(x)\rangle$. By unique continuation, δ and ε are positive on a dense open subset. In fact, $\delta, \varepsilon \geqslant 2$ on a dense open subset. To see this, observe that if $\delta = 1$ on a non-empty open subset, then there is a non-empty open subset $V \subset U$ and a function $f \in C^\infty(V)$ such that $s_1(x) = f(x)s_2(x)$ for every $x \in V$. Therefore, $\sigma(\mathrm{d}f)s_2 = 0$ with σ denoting the symbol of D. Since D is elliptic, f must be constant: a contradiction to s_1 and s_2 being linearly independent. The same argument applies to ε.

If $\rho = 2$, then there exists an $x \in U$ such that $\delta(x) = \varepsilon(x) = 2$; therefore: $\varpi(B)$ does not vanish at x. If $\rho = 3$, then there is an $x \in U$ such that $\min\{\delta(x), \varepsilon(x)\} \geqslant 2$. If $\delta(x) = \varepsilon(x) = 3$, then $\varpi(B)$ evidently does not vanishing at x; otherwise, without loss of generality, $s_1(x)$ and $s_2(x)$ are linearly independent, and $s_3(x) = \lambda_1 s_1(x) + \lambda_2 s_2(x)$. In the latter case,

$$\varpi(B)(x) = s_1(x) \otimes (t_1(x) + \lambda_1 t_3(x)) + s_2(x) \otimes (t_2(x) + \lambda_2 t_3(x))$$

which cannot vanish because $\varepsilon(x) \geqslant 2$. ■

There are examples of first order linear elliptic operators which fail to satisfy Petri's condition up to rank four; see [Wen19b, Example 5.5] or Proposition 2.5.4. Finally, a brief warning: the preceding observation is false when $\mathbf{R}$ is replaced with $\mathbf{C}$ or $\mathbf{H}$. The analogue of Petri's condition only holds up to rank one in this case. (The issue is that $\sigma(\mathrm{d}f)s_2 = 0$ does not imply $\mathrm{d}f = 0$ if f takes values in $\mathbf{C}$ or $\mathbf{H}$.) ♣

1.2 Pulling back and twisting

This section introduces two constructions which produce new linear elliptic operators from old ones: pulling back by a covering map and twisting by a Euclidean local system.

[5] D has the **weak continuation property** if every $s \in \ker D$ which vanishes on an open subset must vanish identically.

[6] The assumptions on D are satisfied provided the coefficients are sufficiently regular, and $D^*D = \nabla^*\nabla +$ lower order terms (and similarly for $D^\dagger$); cf. Remark 1.6.5.

Definition 1.2.1. Let $k \in \mathbf{N}_0$. Let $\pi\colon \tilde{M} \to M$ be a covering map with $\tilde{M}$ connected.[7] Let $D\colon W^{k,2}\Gamma(E) \to L^2\Gamma(F)$ be a linear differential operator of order k. The **pullback** of D by π is the linear differential operator of order k

$$\pi^*D\colon W^{k,2}\Gamma(\pi^*E) \to L^2\Gamma(\pi^*F)$$

characterized by

$$(\pi^*D)(\pi^*s) = \pi^*(Ds).$$ •

Remark 1.2.2. If $\pi\colon \tilde{M} \to M$ is a branched covering map of manifolds whose ramification locus is a closed submanifold of codimension two, then $\tilde{M}$ and M can be equipped with orbifold structures and π can be lifted to an unbranched covering map of orbifolds. Section 2.7 discusses this construction in the case of Riemann surfaces; the higher-dimensional case follows immediately from the two-dimensional case and the above local model. ♣

Definition 1.2.3. A **Euclidean local system on** M is a Euclidean vector bundle $\underline{V}$ over M together with a flat orthogonal connection. •

Remark 1.2.4. Let $x_0 \in M$. Denote by $\pi_1(M, x_0)$ the fundamental group with base-point x_0. If $*$ denotes the usual concatenation of paths, then the multiplication in $\pi_1(M, x_0)$ is defined by $[\gamma_1][\gamma_2] := [\gamma_2 * \gamma_1]$.[8] Parallel transport induces a **monodromy representation**

$$\mu\colon \pi_1(M, x_0) \to \mathrm{O}(V)$$

with V denoting the fiber of $\underline{V}$ over x_0. $\underline{V}$ can be recovered from μ as follows. Denote by $\pi\colon \tilde{M} \to M$ the universal covering map and by $\mathrm{Aut}(\pi)$ the group of deck transformations. A choice of $\tilde{x}_0 \in \pi^{-1}(x_0)$ induces an anti-isomorphism from $\mathrm{Aut}(\pi)$ to $\pi_1(X, x_0)$. Therefore, $\tilde{M}$ is a principal $\pi_1(M, x_0)$–bundle, and $\underline{V}$ is the associated bundle:

$$\underline{V} \cong \tilde{M} \times_\mu V.$$

This sets up a bijection between gauge equivalence classes $[\underline{V}]$ of Euclidean local systems of rank r and equivalence classes $[\mu]$ of representations $\pi_1(M, x_0) \to \mathrm{O}(r)$ up to conjugation by $\mathrm{O}(r)$. For a more detailed discussion—in particular, of how the to interpret the above in the category of orbifolds—we refer the reader to [SY19, Sections 2.4 and 2.5]. ♣

Definition 1.2.5. Let $k \in \mathbf{N}_0$ Let $D\colon W^{k,2}\Gamma(E) \to L^2\Gamma(F)$ be a linear differential operator of order k. Let $\underline{V}$ be a Euclidean local system on M. The **twist** of D by $\underline{V}$ is the linear differential operator of order k

$$D^{\underline{V}}\colon W^{k,2}\Gamma(E \otimes \underline{V}) \to L^2\Gamma(F \otimes \underline{V})$$

characterized as follows: if U is a open subset M, $s \in \Gamma(U, E)$, and $f \in \Gamma(U, \underline{V})$ is covariantly constant with respect to the flat connection on $\underline{V}$, then

$$D^{\underline{V}}(s \otimes f) = (Ds) \otimes f.$$ •

[7] An orbifold map $\pi\colon \tilde{M} \to M$ is a covering map if every point x in the topological space underlying M has a neighborhood of the form U/G with U a G–manifold, $\pi^{-1}(U/G)$ also is of the form $\tilde{U}/G$ with $\tilde{U}$ a G–manifold, and π induces a G–equivariant covering map $\tilde{U} \to U$; see [Moe02, Section 5.3; ALR07, Section 2.2].

[8] This definition might appear backwards. However, it does fit better with the notation of category theory; in particular, it is forced in the definition of the fundamental groupoid $\Pi(M)$.

Proposition 1.2.9 shows that the pullback $\pi^* D$ is equivalent to the twist $D^{\underline{V}}$ for a suitable choice of $\underline{V}$. Its statement requires the following preparation.

Definition 1.2.6. Let $\pi\colon \tilde{M} \to M$ be a finite covering map. Let E be a Euclidean vector bundle over $\tilde{M}$. The **pushforward of** E **by** π is the unique Euclidean vector bundle $\pi_* E$ over X such that for every $x \in X$

$$(\pi_* E)_x = \bigoplus_{\tilde{x} \in \pi^{-1}(x)} E_{\tilde{x}}$$

as Euclidean vector spaces, and such that a section s of $\pi_* E$ is smooth if and only if the induced section $\tilde{s}$ of E is smooth. •

Remark 1.2.7. The following facts about the construction from Definition 1.2.6 are important.

(1) If E is a Euclidean vector bundle over $\tilde{M}$, then the sheaf $\Gamma(\cdot, \pi_* E)$ is the sheaf-theoretic pushforward of the sheaf $\Gamma(\cdot, E)$; that is: there are canonical isomorphisms

$$\Gamma(U, \pi_* E) \cong \Gamma(\pi^{-1}(U), E)$$

for every open set $U \subset M$ and these are compatible with the restriction maps.

(2) If E is a Euclidean local system on $\tilde{M}$, then $\pi_* E$ is a Euclidean local system on M: $s \in \Gamma(U, \pi_* E)$ is covariantly constant if and only if $\tilde{s} \in \Gamma(\pi^{-1}(U), E)$ is.

(3) Let E and F be a Euclidean vector bundles over M and $\tilde{M}$ respectively. For every $x \in M$ there is a conical isomorphism

$$\pi_*(\pi^* E \otimes F)_x \cong \bigoplus_{\tilde{x} \in \pi^{-1}(x)} E_x \otimes F_{\tilde{x}} \cong (E \otimes \pi_* F)_x.$$

These assemble into the **push-pull formula**

$$\pi_*(\pi^* E \otimes F) \cong E \otimes \pi_* F.$$

In particular,

$$\pi_*(\pi^* E) \cong E \otimes \pi_* \underline{\mathbf{R}}.$$

Here $\underline{\mathbf{R}}$ denotes the trivial rank one Euclidean local system on $\tilde{M}$. ♣

Definition 1.2.8. Let G be a group and let $H < G$ be a subgroup. The **normal core** of H is the normal subgroup

$$N \coloneqq \bigcap_{g \in G} gHg^{-1}.$$ •

Proposition 1.2.9. *Let $k \in \mathbf{N}_0$. Let $\pi\colon \tilde{M} \to M$ be a finite covering map with $\tilde{M}$ connected. Let $x_0 \in M$ and $\tilde{x}_0 \in \pi^{-1}(x_0)$. Denote by*

$$C \coloneqq \pi_* \pi_1(\tilde{M}, \tilde{x}_0) < \pi_1(M, x_0)$$

the characteristic subgroup of the covering map and by N the normal core of C. Set

$$S := \pi_1(M, x_0)/C.$$

Denote by $\underline{\mathbf{R}}$ the trivial rank one Euclidean local system on $\tilde{M}$. Set

$$\underline{V} := \pi_* \underline{\mathbf{R}}.$$

Let $D\colon W^{k,2}\Gamma(E) \to L^2\Gamma(F)$ be a linear differential operator of order k. The following hold:

(1) *The monodromy representation of $\underline{V}$ factors through $G := \pi_1(M, x_0)/N$; indeed, it is induced by the representation of G on $\mathrm{Map}(S, \mathbf{R})$ defined by*

$$(\mu_g f)(s) := f(g^{-1}s).$$

(2) *The push-pull formula induces isometries*

$$\pi_*\colon W^{k,2}\Gamma(\pi^*E) \cong W^{k,2}\Gamma(E \otimes \underline{V}) \quad \text{and} \quad \pi_*\colon L^2\Gamma(\pi^*F) \cong L^2\Gamma(F \otimes \underline{V})$$

such that

$$D^{\underline{V}} = \pi_* \circ \pi^* D \circ \pi_*^{-1}.$$

Remark 1.2.10. If π is a normal covering, then $C = N$ and $G = \pi_1(M, x_0)/N$ is isomorphic to its deck transformation group. If π has k sheets, then C has index k. Its normal core has index at most $k!$ by an elementary result known as Poincaré's Theorem. This theorem follows from the observation that the kernel of the canonical homomorphism $\pi_1(M, x_0) \to \mathrm{Bij}(G/C)$ is precisely N and $\mathrm{Bij}(G/C) \cong S_k$. Here $\mathrm{Bij}(G/C)$ denotes the set of bijections of G/C, ♣

Proof of Proposition 1.2.9. The monodromy representation $\mu\colon \pi_1(M, x_0) \to \mathrm{O}(V)$ of $\underline{V}$ is trivial on C; hence, it factors through G. Denote by $\rho\colon (\hat{M}, \hat{x}_0) \to (M, x_0)$ the pointed covering map with characteristic subgroup N. $\hat{M}$ is a principal G–bundle and $\tilde{M} \cong \hat{M} \times_G S$. This implies the assertion about the monodromy representation.

By Remark 1.2.7 (3),

$$\pi_*\pi^*E \cong E \otimes \pi_*\underline{\mathbf{R}} = E \otimes \underline{V}.$$

Denote the resulting isomorphism $\Gamma(\pi^*E) \cong \Gamma(\pi_*\pi^*E) \cong \Gamma(E \otimes \underline{V})$ by π_*. For $s \in \Gamma(E)$ and $f \in C^\infty(\tilde{M})$

$$\pi_*((\pi^*s)f) = s \otimes \pi_* f.$$

Let U be an open subset of M, $s \in \Gamma(U, E)$, and $f \in \Gamma(U, \underline{V})$. Suppose that f is covariantly constant. This is equivalent to the corresponding function $\tilde{f} := (\pi_*)^{-1} f$ on $\tilde{U} := \pi^{-1}(U)$ being locally constant. By the characterizing properties of $D^{\underline{V}}$ and π^*D and since π^*D is a differential operator,

$$D^{\underline{V}}(s \otimes f) = (Ds) \otimes f$$

and

$$(\pi^*D)(\pi_*)^{-1}(s \otimes f) = (\pi^*D)(\pi^*s \cdot \tilde{f}) = \pi^*(Ds) \cdot \tilde{f} = (\pi_*)^{-1}(Ds \otimes f).$$

This proves that $D^{\underline{V}} = \pi_* \circ \pi^* D \circ \pi_*^{-1}$. ■

1.3 Equivariant Brill–Noether loci, I: twists

Pulling back and twisting lead to families of linear elliptic differential operators which fail to satisfy the hypotheses of Theorem 1.1.5 (except for a few corner cases). In this section we formulate a variant of this result which applies to families of twisted linear elliptic differential operators. Throughout this section, assume the following.

Situation 1.3.1. Let $x_0 \in M$. Let $\mathfrak{V} = (\underline{V}_\alpha)_{\alpha=1}^m$ be a finite collection of irreducible Euclidean local systems which are pairwise non-isomorphic. (A Euclidean local system is **irreducible** if it is not a direct sum of two non-zero Euclidean local systems.) For every $\alpha = 1, \dots, m$ denote by $\mathbf{K}_\alpha$ the algebra of parallel endomorphisms of $\underline{V}_\alpha$ and set $k_\alpha := \dim_{\mathbf{R}} \mathbf{K}_\alpha$. ×

Remark 1.3.2. Since $\underline{V}_\alpha$ is irreducible, $\mathbf{K}_\alpha$ is a division algebra; hence, by Frobenius' Theorem it is isomorphic to either $\mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$ and $k_\alpha \in \{1, 2, 4\}$. ♣

If D is a linear elliptic differential operator, then the twists $D^{\underline{V}_\alpha}$ commute with the action of $\mathbf{K}_\alpha$. Therefore, $\ker D^{\underline{V}_\alpha}$ and $\operatorname{coker} D^{\underline{V}_\alpha}$ are left $\mathbf{K}_\alpha$–modules and, hence, right $\mathbf{K}_\alpha^{\mathrm{op}}$–modules. Here $\mathbf{K}_\alpha^{\mathrm{op}}$ denotes the opposite algebra of $\mathbf{K}_\alpha$.

Definition 1.3.3. Let $(D_p)_{p\in\mathscr{P}}$ be a family of linear elliptic differential operators. For $d, e \in \mathbf{N}_0^m$ define the **$\mathfrak{V}$–equivariant Brill–Noether locus** $\mathscr{P}_{d,e}^{\mathfrak{V}}$ by

$$\mathscr{P}_{d,e}^{\mathfrak{V}} := \left\{p \in \mathscr{P} : \dim_{\mathbf{K}_\alpha} \ker D_p^{\underline{V}_\alpha} = d_\alpha \text{ and } \dim_{\mathbf{K}_\alpha} \operatorname{coker} D_p^{\underline{V}_\alpha} = e_\alpha \text{ for every } \alpha = 1, \dots, m\right\}. \quad \bullet$$

Remark 1.3.4. Let $i \in \mathbf{Z}^m$. Let $(D_p)_{p\in\mathscr{P}}$ be a family of linear elliptic operators such that $\operatorname{index}_{\mathbf{K}_\alpha} D_p^{\underline{V}_\alpha} = i_\alpha$ for every $p \in \mathscr{P}$ and $\alpha = 1, \dots, m$. If $\mathscr{P}_{d,e}^{\mathfrak{V}} \neq \varnothing$, then $d - e = i$; in particular: $d_\alpha \geqslant i_\alpha$ and $e_\alpha \geqslant -i_\alpha$.

If M is a manifold, then

$$\operatorname{index}_{\mathbf{K}_\alpha} D_p^{\underline{V}_\alpha} = \operatorname{rk}_{\mathbf{K}_\alpha} \underline{V}_\alpha \cdot \operatorname{index} D_p$$

by the Atiyah–Singer index theorem; therefore, the i_α all have the same sign. If M is an orbifold, there are corrections terms in the index formula which spoil this relation between the indices; see, e.g., Proposition 2.8.6. ♣

Lemma 1.1.4 immediately implies the following.

Theorem 1.3.5. *Let $(D_p)_{p\in\mathscr{P}}$ be a family of linear elliptic differential operators. For $p \in \mathscr{P}$ define $\Lambda_p^{\mathfrak{V}} \colon T_p\mathscr{P} \to \bigoplus_{\alpha=1}^m \operatorname{Hom}_{\mathbf{K}_\alpha}(\ker D_p^{\underline{V}_\alpha}, \operatorname{coker} D_p^{\underline{V}_\alpha})$ by*

$$\Lambda_p^{\mathfrak{V}}(\hat{p}) := \bigoplus_{\alpha=1}^m \Lambda_p^\alpha(\hat{p}) \quad \textit{and} \quad \Lambda_p^\alpha(\hat{p})s := \mathrm{d}_p D^{\underline{V}_\alpha}(\hat{p})s \mod \operatorname{im} D_p^{\underline{V}_\alpha}.$$

Let $d, e \in \mathbf{N}_0^m$ and $p \in \mathscr{P}_{d,e}^{\mathfrak{V}}$. If $\Lambda_p^{\mathfrak{V}}$ is is surjective, then the following hold:

1. *There is an open neighborhood $\mathscr{U}$ of $p \in \mathscr{P}$ such that $\mathscr{P}_{d,e}^{\mathfrak{V}} \cap \mathscr{U}$ is a submanifold of codimension*
$$\operatorname{codim}(\mathscr{P}_{d,e}^{\mathfrak{V}} \cap \mathscr{U}) = \sum_{\alpha=1}^m k_\alpha d_\alpha e_\alpha.$$

(2) $\mathscr{P}^{\mathfrak{V}}_{\tilde{d},\tilde{e}} \neq \varnothing$ *for every* $\tilde{d}, \tilde{e} \in \mathbf{N}_0^m$ *with* $\tilde{d} \leqslant d, \tilde{e} \leqslant e$*, and* $\tilde{d} - \tilde{e} = d - e$. ■

Remark 1.3.6. If E and F are Hermitian vector bundles and $(D_p)_{p\in\mathscr{P}}$ is a family of complex linear elliptic differential operators, then Theorem 1.3.5 does not apply; cf. Remark 1.1.6. Again, this issue is rectified by replacing $\mathbf{R}$ with $\mathbf{C}$ throughout. In fact, this somewhat simplifies the discussion since $\mathbf{C}$ is the unique complex division algebra; hence, there is no need to introduce $\mathbf{K}_\alpha$. ♣

Remark 1.3.7 (Twists by arbitrary Euclidean local systems). Every Euclidean local system $\underline{V}$ decomposes into irreducible local systems

$$\underline{V} \cong \bigoplus_{\alpha=1}^{m} \underline{V}_\alpha^{\oplus \ell_\alpha}$$

with $\ell_1, \ldots, \ell_m \in \mathbf{N}_0$ for a suitable choice of $\mathfrak{V}$. For every $\bar{d}, \bar{e} \in \mathbf{N}_0$ the Brill–Noether locus

$$\mathscr{P}^{\underline{V}}_{\bar{d},\bar{e}} := \left\{ p \in \mathscr{P} : \dim\ker D_p^{\underline{V}} = \bar{d} \text{ and } \dim\operatorname{coker} D_p^{\underline{V}} = \bar{e} \right\}$$

is the finite disjoint union of the subsets $\mathscr{P}^{\mathfrak{V}}_{d,e}$ with $(d,e) \in \mathbf{N}_0^m \times \mathbf{N}_0^m$ satisfying

$$\sum_{\alpha=1}^{m} \ell_\alpha k_\alpha d_\alpha = \bar{d} \quad \text{and} \quad \sum_{\alpha=1}^{m} \ell_\alpha k_\alpha e_\alpha = \bar{e}.$$

Through this observation Theorem 1.3.5 can be brought to bear on families of linear elliptic differential operators twisted by $\underline{V}$. ♣

Definition 1.1.9, Definition 1.1.11, and Proposition 1.1.12 have the following analogues in the present situation.

Definition 1.3.8. A family of linear elliptic differential operators $(D_p)_{p\in\mathscr{P}}$ is **$\mathfrak{V}$–equivariantly flexible in U at $p_\star \in \mathscr{P}$** if for every $A \in \Gamma_c(U, \operatorname{Hom}(E,F))$ there is a $\hat{p} \in T_{p_\star}\mathscr{P}$ such that

$$\mathrm{d}_{p_\star} D^{\underline{V}_\alpha}(\hat{p})s = (A \otimes \mathrm{id}_{\underline{V}_\alpha})s \mod \operatorname{im} D_{p_\star}^{\underline{V}_\alpha}$$

for every $\alpha = 1, \ldots, m$ and $s \in \ker D_{p_\star}^{\underline{V}_\alpha}$. •

Definition 1.3.9. The **$\mathfrak{V}$–equivariant Petri map**

$$\varpi^{\mathfrak{V}} \colon \bigoplus_{\alpha=1}^{m} \Gamma(E \otimes \underline{V}_\alpha) \otimes_{\mathbf{K}_\alpha^{\mathrm{op}}} \Gamma(F^\dagger \otimes \underline{V}_\alpha^*) \to \Gamma(E \otimes F^\dagger)$$

is defined by $\varpi^{\mathfrak{V}} := \sum_{\alpha=1}^{m} \varpi_\alpha$ with ϖ_α denoting the composition of the Petri map

$$\varpi_\alpha \colon \Gamma(E \otimes \underline{V}_\alpha) \otimes_{\mathbf{K}_\alpha^{\mathrm{op}}} \Gamma(F^\dagger \otimes \underline{V}_\alpha^*) \to \Gamma(E \otimes F^\dagger \otimes \underline{V}_\alpha \otimes_{\mathbf{K}_\alpha^{\mathrm{op}}} \underline{V}_\alpha^*)$$

and the map induced by

$$\mathrm{tr} \colon \underline{V}_\alpha \otimes_{\mathbf{K}_\alpha^{\mathrm{op}}} \underline{V}_\alpha^* \to \underline{\mathbf{R}}.$$

Here $\underline{V}_\alpha^* := \mathrm{Hom}(\underline{V}_\alpha, \underline{\mathbf{R}})$ is the dual of $\underline{V}_\alpha$. (Its algebra of parallel endomorphisms is $\mathbf{K}_\alpha^{\mathrm{op}}$.)

Let $U \subset M$ be an open subset. A linear elliptic differential operator $D\colon \Gamma(E) \to \Gamma(F)$ satisfies the **$\mathfrak{V}$–equivariant Petri condition in** U if the map

$$\varpi_{D,U}^{\mathfrak{V}}\colon \bigoplus_{\alpha=1}^{m} \ker D^{\underline{V}_\alpha} \otimes_{\mathbf{K}_\alpha^{\mathrm{op}}} \ker D^{\underline{V}_\alpha,\dagger} \to L^1\Gamma(U, E \otimes F^\dagger)$$

induced by the $\mathfrak{V}$–equivariant Petri map is injective. Here $D^{\underline{V}_\alpha,\dagger} := \left(D^{\underline{V}_\alpha}\right)^\dagger$. •

Remark 1.3.10. The $\mathfrak{V}$–equivariant Petri condition appears even more difficult to verify than the Petri condition. It turns out, however, for the method discussed in Section 1.6 there is no substantial difference between these conditions. ♣

Proposition 1.3.11. *Let $(D_p)_{p\in\mathscr{P}}$ be a family of linear elliptic differential operators. Let $U \subset M$ be an open subset. If $(D_p)_{p\in\mathscr{P}}$ is $\mathfrak{V}$–equivariantly flexible in U at $p_\star \in \mathscr{P}$ and $D_{p_\star}$ satisfies the $\mathfrak{V}$–equivariant Petri condition in U, then the map $\Lambda_{p_\star}^{\mathfrak{V}}$ defined in Theorem 1.3.5 is surjective.* ■

Remark 1.3.12. There are analogues of the observations from Remark 1.1.14 in the equivariant setting.

(1) Every $p \in \mathscr{P}_{d,e}^{\mathfrak{V}}$ has an open neighborhood $\mathscr{U}$ in $\mathscr{P}$ such that $\mathscr{P}_{d,e}^{\mathfrak{V}} \cap \mathscr{U}$ is contained in a submanifold of codimension $\mathrm{rk}\,\Lambda_p^{\mathfrak{V}}$.

(2) Let $\rho \in \mathbf{N}_0^m$ and let $U \subset M$ be an open subset. A linear elliptic differential operator $D\colon W^{k,2}\Gamma(E) \to L^2\Gamma(F)$ satisfies the **$\mathfrak{V}$–equivariant Petri condition up to rank ρ in** U if for every non-zero $B = (B_1, \ldots, B_m) \in \bigoplus_{\alpha=1}^m \ker D_p^{\underline{V}_\alpha} \otimes_{\mathbf{K}_\alpha^{\mathrm{op}}} \ker D_p^{\dagger,\underline{V}_\alpha^*}$ with $\mathrm{rk}\,B_\alpha \leqslant \rho_\alpha$ for $\alpha = 1, \ldots, m$ the section $\varpi^{\mathfrak{V}}(B)$ does not vanish on U. If D_p satisfies this condition and $(D_p)_{p\in\mathscr{P}}$ is $\mathfrak{V}$–equivariantly flexible in U at $p_\star \in \mathscr{P}$, then

$$\mathrm{rk}\,\Lambda_{p_\star} \geqslant \sum_{\alpha=1}^{m} \min\{\rho_\alpha, d_\alpha, e_\alpha\} \max\{d_\alpha, e_\alpha\}.$$

(3) Let $\rho \in \mathbf{N}_0^m$ and let $U \subset M$ be an open subset. Every first order linear elliptic differential operator $D\colon \Gamma(E) \to \Gamma(F)$ satisfies the $\mathfrak{V}$–equivariant Petri condition up to rank ρ on U provided

$$\sum_{\alpha=1}^{m} \mathrm{rk}_{\mathbf{R}}\, V_\alpha \cdot \rho_\alpha \leqslant 3. \tag{1.3.13}$$

Proof. Set $G := \pi_1(M, x_0)$. Denote by $\pi\colon (\tilde{M}, \tilde{x}_0) \to (M, x_0)$ the universal covering map. Every $s \in \ker D^{\underline{V}_\alpha}$ can be regarded as an element $\tilde{s} \in \Gamma(\pi^*E \otimes V_\alpha)^G$ in the space of G–invariant sections, with $G \to \mathrm{O}(V_\alpha)$ denoting the monodromy representation of $\underline{V}_\alpha$. This section can be regarded as $r_\alpha := \mathrm{rk}_{\mathbf{R}}\, V_\alpha$ sections $s_1, \ldots, s_{r_\alpha}$ of π^*E. For every $\alpha = 1, \ldots, m$ let $s_1^\alpha, \ldots, s_{q_\alpha}^\alpha \in \ker D^{\underline{V}_\alpha}$ be linearly independent over $\mathbf{K}_\alpha$. The resulting collection of sections $s^\alpha_{j,k} \in \ker \pi^* D$ ($\alpha = 1, \dots, m$, $j = 1, \dots, q_\alpha$, $k = 1, \dots, r_\alpha$) are linearly independent. The latter is a consequence of Proposition 1.3.14 applied to $W = \ker \pi^* D$. (Analogous statements hold for $D^\dagger$ instead of D.) At this point, one can apply the argument in Remark 1.1.14(3). ■

Unfortunately, this is not as useful as Remark 1.1.14(3) because (1.3.13) is very restrictive; however, it is what lies at the heart of Eftekhary's proof of the 4–rigidity conjecture [Eft16]. ♣

The following discussion is not just needed to justify Remark 1.3.12 (3), but also plays a crucial role in verifying the $\mathfrak{V}$–equivariant Petri condition in Section 1.6.

Let G be a group. For a vector space V with an action of G, denote by V^G the subspace of G–invariant vectors, and set $\mathrm{End}_G(V) := \mathrm{End}(V)^G$ and $\mathrm{Hom}_G(V, W) := \mathrm{Hom}(V, W)^G$ with W a further vector space with an action of G.

Proposition 1.3.14. *Let $(V_\alpha)_{\alpha=1}^m$ be a finite collection of irreducible finite-dimensional orthogonal representations which are pairwise non-isomorphic. Set $\mathbf{K}_\alpha := \mathrm{End}_G(V_\alpha)$. For every representation W of G the map*

$$\mathrm{tr}\colon \bigoplus_{\alpha=1}^m V_\alpha^* \otimes_{\mathbf{K}_\alpha} (V_\alpha \otimes W)^G \to W$$

induced by the trace maps $V_\alpha^* \otimes_{\mathbf{K}_\alpha} V_\alpha \to \mathbf{R}$ is injective.

Proof of Proposition 1.3.14. If tr is not injective, then there are finite-dimensional $\mathbf{K}_\alpha$–linear subspaces $X_\alpha \subset (V_\alpha \otimes W)^G$ such that

$$\mathrm{tr}\colon \bigoplus_{\alpha=1}^m V_\alpha^* \otimes_{\mathbf{K}_\alpha} X_\alpha \to W$$

is not injective. (W need not be finite-dimensional.)

G acts on V_α^* via the contragredient representation and trivially on X_α. Choose a $\mathbf{K}_\alpha$–sesquilinear inner product on X_α (e.g., by choosing a basis $X_\alpha \cong \mathbf{K}_\alpha^{d_\alpha}$). This exhibits $V_\alpha^* \otimes_{\mathbf{K}_\alpha} X_\alpha$ as an orthogonal representation. Since tr is G–equivariant,

$$\ker \mathrm{tr} \subset \bigoplus_{\alpha=1}^m V_\alpha^* \otimes_{\mathbf{K}_\alpha} X_\alpha$$

is an orthogonal subrepresentation.

Since $\ker \mathrm{tr}$ is an orthogonal representation, it decomposes into irreducible orthogonal representations

$$\ker \mathrm{tr} \cong \bigoplus_{\beta=1}^n V_\beta^* \otimes_{\mathbf{K}_\beta} \mathbf{K}_\beta^{d_\beta}.$$

A priori, $(V_\beta)_{\beta=1}^n$ might not be a subset of $(V_\alpha)_{\alpha=1}^m$. However, for every copy of V_β^* appearing in the above decomposition, the induced map

$$V_\beta^* \to \bigoplus_{\alpha=1}^m V_\alpha^* \otimes_{\mathbf{K}_\alpha} X_\alpha$$

is injective. Therefore, by Schur's Lemma, β is among the α. Moreover, the image of the of the above map is $V_\beta^* \otimes_{\mathbf{K}_\beta} L$ for some $L \subset X_\beta$ with $\dim_{\mathbf{K}_\beta} L = 1$. Denote by $S_\beta \subset X_\beta$ the $\mathbf{K}_\beta$–linear subspace spanned by the corresponding lines L. The upshot of this discussion is that

$$\ker \mathrm{tr} = \bigoplus_{\alpha=1}^{m} V_\alpha^* \otimes_{\mathbf{K}_\alpha} S_\alpha.$$

For every non-zero $T \in S_\alpha$ there is a $v^* \in V_\alpha^*$ with $\mathrm{tr}(v^* \otimes T) \neq 0$. Therefore, $S_\alpha = 0$; hence: $\ker \mathrm{tr} = 0$. ∎

1.4 Equivariant Brill–Noether loci, II: pullbacks

In this section we formulate a variant of Theorem 1.1.5 which applies to families of linear elliptic differential operators pulled back by a finite normal covering map. (This is not needed in Part 2.) Throughout this section, assume the following.

Situation 1.4.1. Let $x_0 \in M$. Let G be the quotient of $\pi_1(M, x_0)$ by a finite index normal subgroup N. Denote by $\pi\colon (\tilde{M}, \tilde{x}_0) \to (M, x_0)$ a pointed covering map with characteristic subgroup N. Let

$$\mu_\alpha\colon G \to \mathrm{O}(V_\alpha) \quad (\alpha = 1, \dots, m = m(G))$$

be the irreducible representations of G (up to isomorphism). Set

$$\mathbf{K}_\alpha := \mathrm{End}_G(V_\alpha) \quad \text{and} \quad k_\alpha := \dim_{\mathbf{R}} \mathbf{K}_\alpha. \qquad \times$$

If $D\colon W^{k,2}\Gamma(E) \to L^2\Gamma(F)$ is a linear elliptic differential operator of order k, then $\ker \pi^* D$ and $\operatorname{coker} \pi^* D$ are representations of G. Every representation V of G can be decomposed into irreducible representations. The evaluation map defines a G–equivariant isomorphism

$$\mathrm{ev}\colon \bigoplus_{\alpha=1}^{m} \mathrm{Hom}_G(V_\alpha, V) \otimes_{\mathbf{K}_\alpha} V_\alpha \cong V. \tag{1.4.2}$$

Hence,

$$V \cong \bigoplus_{\alpha=1}^{m} V_\alpha^{\oplus d_\alpha} \quad \text{with} \quad d_\alpha := \dim_{\mathbf{K}_\alpha^{\mathrm{op}}} \mathrm{Hom}_G(V_\alpha, V).$$

In particular, $d = (d_1, \dots, d_m) \in \mathbf{N}^m$ determines V up to isomorphism.

Definition 1.4.3. Let $(D_p)_{p \in \mathscr{P}}$ be a family of linear elliptic differential operators. For $d, e \in \mathbf{N}_0^m$ define the G**–equivariant Brill–Noether locus** $\mathscr{P}_{d,e}^G$ by

$$\mathscr{P}_{d,e}^G := \left\{ p \in \mathscr{P} : \begin{array}{l} \dim_{\mathbf{K}_\alpha^{\mathrm{op}}} \mathrm{Hom}_G(V_\alpha, \ker \pi^* D_p) = d_\alpha \text{ and} \\ \dim_{\mathbf{K}_\alpha^{\mathrm{op}}} \mathrm{Hom}_G(V_\alpha, \operatorname{coker} \pi^* D_p) = e_\alpha \text{ for every } \alpha = 1, \dots, m \end{array} \right\}. \qquad \bullet$$

Remark 1.4.4. Let $D\colon \Gamma(E) \to \Gamma(F)$ is a linear elliptic differential operator. The G–equivariant index of π^*D is $\operatorname{index}_G \pi^*D := [\ker \pi^*D] - [\operatorname{coker} \pi^*D] \in R(G)$. Here $R(G)$ denotes the representation ring of G; its elements are formal differences of isomorphism classes of representations of G. It is a consequence of the above discussion that $R(G) \cong \mathbf{Z}^m$ as abelian groups.

For families of linear elliptic operators with G–equivariant index corresponding to $i \in \mathbf{Z}^m$ what was said in Remark 1.3.4 applies in the present situation as well. ♣

Lemma 1.1.4 has the following refinement for G–equivariant Fredholm operators.

Lemma 1.4.5. *Let X and Y be two Banach spaces equipped with G–actions. Denote by $\mathscr{F}_G(X,Y)$ the space of G–equivariant Fredholm operators. For every $L \in \mathscr{F}_G(X,Y)$ there is an open neighborhood $\mathscr{U} \subset \mathscr{F}_G(X,Y)$ and a smooth map $\mathscr{S}\colon \mathscr{U} \to \operatorname{Hom}_G(\ker L, \operatorname{coker} L)$ such that for every $T \in \mathscr{U}$ there are G–equivariant isomorphisms*

$$\ker T \cong \ker \mathscr{S}(T) \quad \textit{and} \quad \operatorname{coker} T \cong \operatorname{coker} \mathscr{S}(T);$$

furthermore, $\mathrm{d}_L\mathscr{S}\colon T_L\mathscr{F}_G(X,Y) \to \operatorname{Hom}_G(\ker L, \operatorname{coker} L)$ satisfies

$$\mathrm{d}_L\mathscr{S}(\hat{L})s = \hat{L}s \mod \operatorname{im} L.$$

Proof. The proof of Lemma 1.1.4 carries over provided $\operatorname{coim} L$ and the lift of $\operatorname{coker} L$ are chosen G–invariant. ■

Lemma 1.4.5 immediately implies the following.

Theorem 1.4.6. *Let $(D_p)_{p\in\mathscr{P}}$ be a family of linear elliptic differential operators. For $p \in \mathscr{P}$ define $\Lambda_p^G\colon T_p\mathscr{P} \to \operatorname{Hom}_G(\ker \pi^*D_p, \operatorname{coker} \pi^*D_p)$ by*

$$\Lambda_p^G(\hat{p})s := \mathrm{d}_p(\pi^*D)(\hat{p})s \mod \operatorname{im} \pi^*D_p.$$

Let $d, e \in \mathbf{N}_0^m$ and $p \in \mathscr{P}_{d,e}^G$. If Λ_p^G is surjective, then the following hold:

(1) *There is an open neighborhood $\mathscr{U}$ of $p \in \mathscr{P}$ such that $\mathscr{P}_{d,e}^G \cap \mathscr{U}$ is a submanifold of codimension*
$$\operatorname{codim}(\mathscr{P}_{d,e}^G \cap \mathscr{U}) = \sum_{\alpha=1}^m k_\alpha d_\alpha e_\alpha.$$

(2) *$\mathscr{P}_{\tilde{d},\tilde{e}}^G \neq \varnothing$ for every $\tilde{d}, \tilde{e} \in \mathbf{N}_0^m$ with $\tilde{d} \leqslant d$, $\tilde{e} \leqslant e$, and $\tilde{d} - \tilde{e} = d - e$.* ■

Remark 1.4.7 (Pullbacks by arbitrary covering maps). Suppose that $\pi\colon \tilde{M} \to M$ is a finite covering map with characteristic subgroup $C < \pi_1(M, x_0)$. Denote by N the normal core of C, denote by $\rho\colon (\hat{M}, \hat{x}_0) \to (M, x_0)$ the pointed covering map with characteristic subgroup N, and set $G := \pi_1(M, x_0)/N$. For $S := \pi_1(M, x_0)/C$ the decomposition (1.4.2) of $\operatorname{Map}(S, \mathbf{R})$ is

$$\operatorname{Map}(S, \mathbf{R}) \cong \bigoplus_{\alpha=1}^m (V_\alpha^*)^C \otimes_{\mathbf{K}_\alpha} V_\alpha;$$

indeed: the map $\mathrm{ev}_{[1]}\colon\ \mathrm{Hom}_G(V_\alpha, \mathrm{Map}(S, \mathbf{R})) \to V_\alpha^*$ defined by $\mathrm{ev}_1(\ell)(v) := \ell(v)([1])$ is injective and its image is $(V_\alpha^*)^C$. Therefore, by Proposition 1.2.9(2)

$$\ker \pi^* D \cong \bigoplus_{\alpha=1}^m V_\alpha^C \otimes_{\mathbf{K}_\alpha^{\mathrm{op}}} \mathrm{Hom}_G(V_\alpha, \ker \rho^* D) \quad \text{and}$$
$$\operatorname{coker} \pi^* D \cong \bigoplus_{\alpha=1}^m V_\alpha^C \otimes_{\mathbf{K}_\alpha^{\mathrm{op}}} \mathrm{Hom}_G(V_\alpha, \operatorname{coker} \rho^* D).$$

With the above in mind Theorem 1.4.6 can be brought to bear on non-normal covering maps; cf. Remark 1.3.7 ♣

Definition 1.1.9, Definition 1.1.11, and Proposition 1.1.12 have the following analogues in the present situation.

Definition 1.4.8. A family of linear elliptic differential operators $(D_p)_{p\in\mathscr{P}}$ is G**–equivariantly flexible in** U **at** $p_\star \in \mathscr{P}$ if for every $A \in \Gamma(\mathrm{Hom}(E, F))$ supported in U there is a $\hat{p} \in T_{p_\star}\mathscr{P}$ such that

$$\mathrm{d}_{p_\star}(\pi^* D)(\hat{p})s = (\pi^* A)s \mod \operatorname{im} \pi^* D_{p_\star}$$

for every $s \in \ker \pi^* D_{p_\star}$. •

Definition 1.4.9. Let $U \subset M$ be an open subset. A linear elliptic differential operator $D\colon\ W^{k,2}\Gamma(E) \to L^2\Gamma(F)$ satisfies the G**–equivariant Petri condition in** U if the map

$$\varpi_{D,U}^G\colon\ (\ker \pi^* D \otimes \ker \pi^* D^\dagger)^G \to \Gamma(\pi^{-1}(U), \pi^* E \otimes \pi^* F^\dagger)^G$$

induced by the Petri map is injective. •

Remark 1.4.10. Remark 1.3.10 applies to the the G–equivariant Petri condition as well. ♣

Proposition 1.4.11. *Let $(D_p)_{p\in\mathscr{P}}$ be a family of linear elliptic differential operators. Let $U \subset M$ be an open subset. If $(D_p)_{p\in\mathscr{P}}$ is G–equivariantly flexible in U at $p_\star \in \mathscr{P}$ and $D_{p_\star}$ satisfies the G–equivariant Petri condition in U, then the map $\Lambda_{p_\star}^G$ defined in Theorem 1.4.6 is surjective.* ■

1.5 Equivariant Brill–Noether loci, III: comparison

This section discusses the relation between Section 1.3 and Section 1.4. (This is not needed in Part 2.) Throughout this section, assume Situation 1.4.1. This yields an instance of Situation 1.3.1 by setting

$$\underline{V}_\alpha := \tilde{M} \times_{\mu_\alpha} V_\alpha.$$

Denote by $\sigma \in S_m$ the permutation such that $V_\alpha^* \cong V_{\sigma(\alpha)}$. The following summarizes the what lies at the heart of the relation.

Proposition 1.5.1. *Let $D\colon\ W^{k,2}\Gamma(E) \to L^2\Gamma(F)$ be a linear differential operator of order k. The following hold:*

(1) *The action of G by deck transformations of π induces a G–action on the local system*

$$\underline{V} \coloneqq \pi_* \underline{\mathbf{R}}.$$

The isomorphisms $\pi_*\colon\ W^{k,2}\Gamma(\pi^*E) \cong W^{k,2}\Gamma(E \otimes \underline{V})$ and $\pi_*\colon\ L^2\Gamma(\pi^*F) \cong L^2\Gamma(F \otimes \underline{V})$ from Proposition 1.2.9(2) are G–equivariant.

(2) *There is a G–equivariant isomorphism*

$$\phi\colon\ \underline{V} \cong \bigoplus_{\alpha=1}^{m} V_\alpha^* \otimes_{\mathbf{K}_\alpha} \underline{V}_\alpha.$$

Here G acts on V_α^*.

(3) *Denote by*

$$\psi_E\colon\ W^{k,2}\Gamma(\pi^*E) \cong W^{k,2}\Gamma(E \otimes \underline{V}) \cong \bigoplus_{\alpha=1}^{m} V_\alpha^* \otimes_{\mathbf{K}_\alpha} W^{k,2}\Gamma(E \otimes \underline{V}_\alpha)$$

the G–equivariant isomorphisms induced by π_* and ϕ (and analogously for F and $F^\dagger$). The composition

$$\psi_F \circ \pi^* D \circ \psi_E^{-1}$$

agrees with

$$\bigoplus_{\alpha=1}^{m} \mathrm{id}_{V_\alpha^*} \otimes_{\mathbf{K}_\alpha} D^{\underline{V}_\alpha}\colon\ \bigoplus_{\alpha=1}^{m} V_\alpha^* \otimes_{\mathbf{K}_\alpha} W^{k,2}\Gamma(E \otimes \underline{V}_\alpha) \to \bigoplus_{\alpha=1}^{m} V_\alpha^* \otimes_{\mathbf{K}_\alpha} L^2\Gamma(F \otimes \underline{V}_\alpha).$$

(4) *Define γ to be the composition of the isomorphisms*

$$\begin{gathered}
\Big(\big(\textstyle\bigoplus_{\alpha=1}^{m} V_\alpha^* \otimes_{\mathbf{K}_\alpha} W^{k,2}\Gamma(E \otimes \underline{V}_\alpha)\big) \otimes \big(\textstyle\bigoplus_{\beta=1}^{m} V_\beta^* \otimes_{\mathbf{K}_\beta^{\mathrm{op}}} L^2\Gamma(F^\dagger \otimes \underline{V}_\beta)\big)\Big)^G \\
\Big\downarrow \cong \\
\textstyle\bigoplus_{\alpha,\beta=1}^{m} W^{k,2}\Gamma(E \otimes \underline{V}_\alpha) \otimes_{\mathbf{K}_\alpha^{\mathrm{op}}} (V_\alpha^* \otimes V_{\sigma(\beta)})^G \otimes_{\mathbf{K}_\beta^{\mathrm{op}}} L^2\Gamma(F^\dagger \otimes \underline{V}_{\sigma(\beta)}^*) \\
(\star)\Big\downarrow \cong \\
\textstyle\bigoplus_{\alpha=1}^{m} W^{k,2}\Gamma(E \otimes \underline{V}_\alpha) \otimes_{\mathbf{K}_\alpha^{\mathrm{op}}} L^2\Gamma(F^\dagger \otimes \underline{V}_\alpha^*).
\end{gathered}$$

Here $(\star)$ is induced by the identification

$$(V_\alpha^* \otimes V_{\sigma(\beta)})^G = \begin{cases} \mathbf{K}_\alpha^{\mathrm{op}} & \text{if } \alpha = \sigma(\beta) \\ 0 & \text{otherwise.} \end{cases}$$

The following diagram commutes:

$$\begin{tikzcd}
\big(W^{k,2}\Gamma(\pi^*E) \otimes L^2\Gamma(\pi^*F^\dagger)\big)^G \arrow[r, "\varpi"] \arrow[d, "\gamma\circ(\psi_E \otimes \psi_{F^\dagger})"', "\cong"] & L^1\Gamma(\pi^*(E \otimes F^\dagger))^G \\
\bigoplus_{\alpha=1}^{m} W^{k,2}\Gamma(E \otimes \underline{V}_\alpha) \otimes_{\mathbf{K}_\alpha^{\mathrm{op}}} L^2\Gamma(F^\dagger \otimes \underline{V}_\alpha^*) \arrow[r, "\varpi^{\mathfrak{B}}"'] & L^1\Gamma(E \otimes F^\dagger). \arrow[u, "\cong"', "\pi^*"]
\end{tikzcd} \tag{1.5.2}$$

Proof. Let $g \in G$. Denote by δ_g the corresponding deck transformation: $\delta_g(\tilde{x}) := \tilde{x}g^{-1}$. There is a canonical isomorphism $\underline{\mathbf{R}} \cong (\delta_g)_*\underline{\mathbf{R}}$ identifying $\underline{\mathbf{R}}_{\tilde{x}} = \mathbf{R} = \big((\delta_g)_*\underline{\mathbf{R}}\big)_{\tilde{x}} = \underline{\mathbf{R}}_{\tilde{x}g}$. This defines an isomorphism

$$\underline{V} = \pi_*\underline{\mathbf{R}} \cong \pi_*(\delta_g)_*\underline{\mathbf{R}} = \pi_*\underline{\mathbf{R}} \cong \underline{V}. \tag{1.5.3}$$

A moment's thought shows that this isomorphism maps $v \in \underline{V}_x = C^\infty(\pi^{-1}(x), \mathbf{R})$ to $gv \in \underline{V}_x$ defined by $(gv)(\tilde{x}) := v(\tilde{x}g)$. These isomorphisms (1.5.3) assemble into a G–action on $\underline{V}$. This description makes (1) evident.

The left and right regular representations of G on $\mathbf{R}[G] := \mathrm{Map}(G, \mathbf{R})$ are defined by

$$(\lambda_g f)(x) := f(g^{-1}x) \quad \text{and} \quad (\rho_h f)(x) := f(xh)$$

respectively. By Proposition 1.2.9(1), the monodromy representation of $\underline{V}$ is λ; that is: $\underline{V} \cong \tilde{M} \times_\lambda \mathbf{R}[G]$. Since λ and ρ commute, ρ defines an action of G on $\underline{V}$. This is precisely the action described above.

Since λ_g and ρ_h commute, $(g, h) \mapsto \lambda_g \circ \rho_h$ defines a representation of $G \times G$ on $\mathbf{R}[G]$. $G \times G$ also acts on $V_\alpha^* \otimes_{\mathbf{K}_\alpha} V_\alpha$ via $(g, h) \mapsto \mu_\alpha(h^{-1})^* \otimes \mu_\alpha(g)$. The isomorphism (1.4.2) corresponding to λ reads

$$\bigoplus_{\alpha=1}^m \mathrm{Hom}_G(V_\alpha, \mathbf{R}[G]) \otimes_{\mathbf{K}_\alpha} V_\alpha \cong \mathbf{R}[G].$$

$\mathrm{Hom}_G(V_\alpha, \mathbf{R}[G])$ inherits a G–action from ρ. The map $\mathrm{ev}_{\mathbf{1}} \colon \mathrm{Hom}_G(V_\alpha, \mathbf{R}[G]) \to V_\alpha^*$ defined by $\mathrm{ev}_{\mathbf{1}}(\ell)(v) := \ell(v)(1)$ is a G–equivariant isomorphism. This yields the $G \times G$–equivariant Peter–Weyl isomorphism

$$\psi \colon \mathbf{R}[G] \cong \bigoplus_{\alpha=1}^m V_\alpha^* \otimes_{\mathbf{K}_\alpha} V_\alpha. \tag{1.5.4}$$

It induces a G–equivariant isomorphism

$$\underline{V} \cong \bigoplus_{\alpha=1}^m V_\alpha^* \otimes_{\mathbf{K}_\alpha} \underline{V}_\alpha. \tag{1.5.5}$$

This proves (2).

(2) and Proposition 1.2.9(2) imply (3).

It suffices to prove (4) for $M = \{1\}$ and $\tilde{M} = G$. In this case E and $F^\dagger$ are vector spaces, $\Gamma(\pi^*E) = \mathrm{Map}(G, E) = \mathbf{R}[G] \otimes E$ with the G–action induced by ρ, $\Gamma(E \otimes \underline{V}_\alpha) = E \otimes V_\alpha$ (and analogously for $F^\dagger$ and $E \otimes F^\dagger$). The diagram (1.5.2) becomes

$$\begin{tikzcd}
\big(\mathbf{R}[G] \otimes E \otimes \mathbf{R}[G] \otimes F^\dagger\big)^G \arrow[r, "\varpi"] \arrow[d, "\gamma\circ(\psi_E\otimes\psi_{F^\dagger})"', "\cong"] & (\mathbf{R}[G] \otimes E \otimes F^\dagger)^G \\
\bigoplus_{\alpha=1}^m (E \otimes V_\alpha) \otimes_{\mathbf{K}_\alpha^{\mathrm{op}}} (F^\dagger \otimes V_\alpha^*) \arrow[r, "\varpi^{\mathfrak{B}}"'] & E \otimes F^\dagger. \arrow[u, "\cong", "\pi^*"']
\end{tikzcd} \tag{1.5.6}$$

Since every map in this diagram has a factor $\mathrm{id}_{E\otimes F^\dagger}$, it suffices to prove that it commutes for $E = F^\dagger = \mathbf{R}$.

The map $\varpi\colon (\mathbf{R}[G]\otimes \mathbf{R}[G])^G \to (\mathbf{R}[G])^G$ is the pointwise multiplication and the map $(\pi^*)^{-1}\colon \mathbf{R}[G]^G \to \mathbf{R}$ is evaluation at $\mathbf{1}$. Therefore,

$$(\pi^*)^{-1}\circ\varpi\left(\sum\nolimits_{g,h\in G} a_{g,h}\cdot g\otimes h\right) = a_{\mathbf{1},\mathbf{1}}.$$

The computation of the composition $\varpi^{\mathfrak{B}}\circ\gamma\circ(\psi\otimes\psi)$ relies on the following.

Proposition 1.5.7. *After identifying $V_\alpha^*\otimes_{\mathbf{K}_\alpha} V_\alpha = \mathrm{End}_{\mathbf{K}_\alpha}(V_\alpha)$, the Peter–Weyl isomorphism (1.5.4) is given by*

$$\psi(g) = \bigoplus_{\alpha=1}^m \frac{\dim_{\mathbf{K}_\alpha} V_\alpha}{|G|}\cdot\mu_\alpha(g).$$

Proof. The inverse of the evaluation map $\mathrm{ev}\colon \bigoplus_{\alpha=1}^m \mathrm{Hom}_G(V_\alpha, V)\otimes_{\mathbf{K}_\alpha} V_\alpha \cong V$ is the map $\Pi = (\Pi_1,\dots,\Pi_m)$ with

$$\Pi_\alpha(v) \coloneqq \frac{\dim_{\mathbf{K}_\alpha} V_\alpha}{|G|}\sum_{i=1}^{r_\alpha}\left(\sum_{g\in G}\mu_\alpha^*(g)e_{\alpha,i}^*\otimes\mu(g)v\right)\otimes e_{\alpha,i}.$$

Here $r_\alpha \coloneqq \dim_{\mathbf{R}} V_\alpha$, $e_{\alpha,i}$ ($i=1,\dots,r_\alpha$) is a basis of V_α, and $e_{\alpha,i}^*$ ($i=1,\dots,r_\alpha$) is the dual basis of V_α^*. Indeed,

$$\begin{aligned}
\mathrm{ev}\circ\Pi(v) &= \sum_{\alpha=1}^m \frac{\dim_{\mathbf{K}_\alpha} V_\alpha}{|G|}\sum_{g\in G}\mathrm{tr}(\mu_\alpha(g^{-1}))\cdot\mu(g)v\\
&= \frac{1}{|G|}\sum_{\alpha=1}^m\sum_{g\in G}\mathrm{tr}(\lambda(g^{-1}))\cdot\mu(g)v\\
&= v.
\end{aligned}$$

Here the the first identity follows by direct inspection, the second uses the existence of the Peter–Weyl isomorphism (1.5.4), and the last identity follows by direct computation of $\mathrm{tr}\circ\lambda$. (The composition of Π_α with $\mathrm{ev}_\alpha\colon \mathrm{Hom}_G(V_\alpha,V)\otimes_{\mathbf{K}_\alpha} V_\alpha \to V$ is the projection to the V_α–isotypic component.)

The Peter–Weyl isomorphism (1.5.4) is the composition

$$\psi\colon \mathbf{R}[G] \xrightarrow{\Pi} \bigoplus_{\alpha=1}^m \mathrm{Hom}_G(V_\alpha,\mathbf{R}[G])\otimes_{\mathbf{K}_\alpha} V_\alpha \xrightarrow{\bigoplus_{\alpha=1}^m \mathrm{ev}_{\mathbf{1}}\otimes\mathrm{id}_{V_\alpha}} \bigoplus_{\alpha=1}^m V_\alpha^*\otimes_{\mathbf{K}_\alpha} V_\alpha.$$

By direct computation

$$\psi(g) = \left(\frac{\dim_{\mathbf{K}_\alpha} V_\alpha}{|G|}\sum_{i=1}^{r_\alpha}\mu_\alpha^*(g^{-1})e_{\alpha,i}^*\otimes e_{\alpha,i} : \alpha = 1,\dots,m\right).$$

This yields the asserted expression for ψ. ∎

In the definition of γ, the map $(V_\alpha^* \otimes V_\alpha)^G \to \mathbf{K}_\alpha$ in ($\star$) is induced by the composition

$$V_\alpha^* \otimes V_\alpha \to V_\alpha^* \otimes_{\mathbf{K}_\alpha} V_\alpha \xrightarrow{\frac{\mathrm{tr}_{\mathbf{K}_\alpha}}{\dim_{\mathbf{K}_\alpha} V_\alpha}} \mathbf{K}_\alpha.$$

Therefore, γ is induced by $1/\dim_{\mathbf{K}_\alpha} V_\alpha$ times the map $\mathrm{End}_{\mathbf{K}_\alpha}(V_\alpha) \otimes \mathrm{End}_{\mathbf{K}_\alpha^{\mathrm{op}}}(V_\alpha^*) \to \mathrm{End}_{\mathbf{K}_\alpha}(V_\alpha)$, $A \otimes B \mapsto A \circ B^*$. The Petri map $\varpi^{\mathfrak{B}}$ is the sum of the traces $\mathrm{tr}\colon\ \mathrm{End}_{\mathbf{K}_\alpha}(V_\alpha) \to \mathbf{R}$. Therefore,

$$\begin{aligned} \varpi^{\mathfrak{B}} \circ \gamma \circ (\psi \otimes \psi)\Big(\textstyle\sum_{g,h\in G} a_{g,h} \cdot g \otimes h\Big) &= \sum_{g,h\in G} a_{g,h} \sum_{\alpha=1}^{m} \frac{\dim_{\mathbf{K}_\alpha} V_\alpha}{|G|^2} \cdot \mathrm{tr}(\mu_\alpha(gh^{-1})) \\ &= \frac{1}{|G|} \sum_{g\in G} a_{g,g} \\ &= a_{1,1}. \end{aligned}$$

Here the second identity follows as in the proof of Proposition 1.5.7 above, and the third identity uses the G–invariance: $a_{g,g} = a_{1,1}$. ■

With Proposition 1.5.1 in hand the discussions in Section 1.3 and Section 1.4 can be related as follows:

(1) By Proposition 1.5.1(3), For every $\alpha = 1, \ldots, m$ the isomorphisms ψ_E and ψ_F induce isomorphisms

$$\mathrm{Hom}_G(V_\alpha, \ker \pi^* D) \cong \ker D^{\underline{V}_\alpha^*} \quad \text{and} \quad \mathrm{Hom}_G(V_\alpha, \operatorname{coker} \pi^* D) \cong \operatorname{coker} D^{\underline{V}_\alpha^*}$$

(and analogously for $\pi^* D^\dagger$ and $D^{\underline{V}_\alpha,\dagger}$). If V and W are representations of G, then (1.4.2) induces isomorphisms

$$\begin{aligned} \mathrm{Hom}_G(V, W) &\cong \bigoplus_{\alpha=1}^{m} \mathrm{Hom}_{\mathbf{K}_\alpha}(\mathrm{Hom}_G(V_\alpha^*, V), \mathrm{Hom}_G(V_\alpha^*, W)) \quad \text{and} \\ (V \otimes W)^G &\cong \bigoplus_{\alpha=1}^{m} \mathrm{Hom}_G(V_\alpha^*, V) \otimes_{\mathbf{K}_\alpha^{\mathrm{op}}} \mathrm{Hom}_G(V_\alpha, W). \end{aligned}$$

Hence, there are isomorphisms

$$\begin{aligned} \eta\colon\ \mathrm{Hom}_G(\ker \pi^* D, \operatorname{coker} \pi^* D) &\to \bigoplus_{\alpha=1}^{m} \mathrm{Hom}_{\mathbf{K}_\alpha}(\ker D^{\underline{V}_\alpha}, \operatorname{coker} D^{\underline{V}_\alpha}) \quad \text{and} \\ \tau\colon\ (\ker \pi^* D \otimes \ker \pi^* D^\dagger)^G &\to \bigoplus_{\alpha=1}^{m} \ker D^{\underline{V}_\alpha} \otimes_{\mathbf{K}_\alpha^{\mathrm{op}}} \ker D^{\underline{V}_\alpha,\dagger}. \end{aligned}$$

(2) In the situation of Definition 1.3.3 and Definition 1.4.3,

$$\mathscr{P}^G_{d,e} = \mathscr{P}^{\mathfrak{B}}_{\sigma^* d, \sigma^* e}$$

with $(\sigma^* d)_\alpha = d_{\sigma(\alpha)}$ and $(\sigma^* e)_\alpha = e_{\sigma(\alpha)}$.

(3) In the situation of Theorem 1.3.5 and Theorem 1.4.6,

$$\Lambda_p^{\mathfrak{V}} = \eta \circ \Lambda_p^G.$$

(4) In the situation of Definition 1.4.8, the maps

$$\begin{aligned} \mathrm{ev}_p^{\mathfrak{V}} \colon \Gamma_c(U, \mathrm{Hom}(E,F)) &\to \bigoplus_{\alpha=1}^m \mathrm{Hom}_{\mathbf{K}_\alpha}(\ker D_p^{\underline{V}_\alpha}, \operatorname{coker} D_p^{\underline{V}_\alpha}) \quad \text{and} \\ \mathrm{ev}_p^G \colon \Gamma_c(U, \mathrm{Hom}(E,F)) &\to \mathrm{Hom}_G(\ker \pi^* D_p, \operatorname{coker} \pi^* D_p) \end{aligned}$$

defined by

$$\mathrm{ev}_p^{\mathfrak{V}} := \bigoplus_{\alpha=1}^m \mathrm{ev}_p^\alpha \quad \text{with} \quad \mathrm{ev}_p^\alpha(A)s := (A \otimes \mathrm{id}_{\underline{V}_\alpha})s \mod \operatorname{im} D_p^{\underline{V}_\alpha} \quad \text{and}$$

$$\mathrm{ev}_p^G(A)s := (\pi^* A)s \mod \operatorname{im} \pi^* D_p$$

satisfy

$$\mathrm{ev}_p^{\mathfrak{V}} = \eta \circ \mathrm{ev}_p^G.$$

Therefore, $(D_p)_{p \in \mathscr{P}}$ is G–equivariantly flexible in U at $p \in \mathscr{P}$ if and only if it is $\mathfrak{V}$–equivariantly flexible in U at p.

(5) By Proposition 1.5.1(4), in the situation of Definition 1.4.9, the map $\varpi_{D,U}^G$ satisfies

$$\varpi_{D,U}^G = \pi^* \circ \varpi_{D,U}^{\mathfrak{V}} \circ \tau. \tag{1.5.8}$$

Therefore, D satisfies the G–equivariant Petri condition in U if and only if it satisfies the $\mathfrak{V}$–equivariant Petri condition in U.

1.6 Petri's condition revisited

While Petri's condition typically is hard to verify for any particular elliptic operator, one can sometimes prove that it is satisfied for a generic element of a family of operators. Theorem 1.6.17 provides a useful tool for proving such statements. This result has been developed by Wendl [Wen19b, Section 5.2] and was the essential innovation which allowed Wendl to prove the super-rigidity conjecture.

Throughout this section, let $x \in M$ and, furthermore, amend Definition 1.1.1 as follows.

Definition 1.6.1. Let $k \in \mathbf{N}_0$. A **family of linear elliptic differential operators of order** k **with smooth coefficients** is a family of linear elliptic differential operators $(D_p)_{p \in \mathscr{P}}$ of the form

$$D(p) = \sum_{\ell=0}^k a_\ell(p, \cdot)\nabla^\ell$$

with a_ℓ a smooth section of $\mathrm{pr}_2^* \mathrm{Hom}(T^*M^{\otimes \ell} \otimes E, F)$ over $\mathscr{P} \times M$ ($\ell = 0, \ldots, k$). •

Let us begin by introducing the following algebraic variant of Petri's condition.

Definition 1.6.2. Denote by $\mathscr{E}$ the sheaf of sections of E and by $\mathscr{E}_x$ its stalk at x; that is:

$$\mathscr{E}_x \coloneqq \varinjlim_{x \in U} \Gamma(U, E).$$

If $s \in \mathscr{E}_x$ vanishes at x, then its derivative at x does not depend on the choice of a local trivialization and defines an element $\mathrm{d}_x s \in \mathrm{Hom}(T_x M, E_x)$. If $\mathrm{d}_x s = 0$, then s has a second derivative $\mathrm{d}_x^2 s \in \mathrm{Hom}(S^2 T_x M, E_x)$ at x. Here $S^j T_x M$ is the j–th symmetric tensor power. In general, if $s(x), \mathrm{d}_x s, \dots, \mathrm{d}_x^{j-1} s$ vanish, then s is said to vanish to $(j-1)^{\text{st}}$ order and its j^{th} **derivative**

$$\mathrm{d}_x^j s \in \mathrm{Hom}(S^j T_x M, E_x)$$

is defined. The **vanishing order filtration** $\mathscr{V}_\bullet \mathscr{E}_x$ on $\mathscr{E}_x$ is defined by

$$\mathscr{V}_j \mathscr{E}_x \coloneqq \{s \in \mathscr{E}_x : s \text{ vanishes to } (j-1)^{\text{st}} \text{ order}\}$$

for $j \in \mathbf{N}_0$ and $\mathscr{V}_{-j} \mathscr{E}_x \coloneqq \mathscr{E}_x$ for $j \in \mathbf{N}$. For $\ell \in \mathbf{N}_0$ the **ℓ–jet space of E at x** is

$$J_x^\ell E \coloneqq \mathscr{E}_x / \mathscr{V}_{\ell+1} \mathscr{E}_x.$$

The **∞–jet space of E at x** is

$$J_x^\infty E \coloneqq \varprojlim J_x^\ell E = \mathscr{E}_x / \mathscr{V}_\infty \mathscr{E}_x \quad \text{with} \quad \mathscr{V}_\infty \mathscr{E}_x \coloneqq \bigcap_{j \in \mathbf{Z}} \mathscr{V}_j \mathscr{E}_x.$$

For $\ell \in \mathbf{N}_0 \cup \{\infty\}$ the **ℓ–jet** of a linear differential operator $D\colon \Gamma(E) \to \Gamma(F)$ of order k with smooth coefficients is the linear map

$$J_x^\ell D\colon J_x^{k+\ell} E \to J_x^\ell F.$$

induced by D. •

Definition 1.6.3. The ∞–jet of a linear elliptic differential operator $J_x^\infty D\colon J_x^\infty E \to J_x^\infty F$ satisfies the **∞–jet Petri condition** if the map

$$\varpi_{J_x^\infty D}\colon \ \ker J_x^\infty D \otimes \ker J_x^\infty D^\dagger \to J_x^\infty (E \otimes F^\dagger)$$

induced by the Petri map is injective. •

The ∞–jet Petri condition and the equivariant Petri conditions are related by the following proposition.

Definition 1.6.4. Let $x \in M$. Let U be an open neighborhood of $x \in M$. A differential operator $D\colon \Gamma(E) \to \Gamma(F)$ has the **strong unique continuation property at x in U** if the map

$$\ker(D\colon \Gamma(U, E) \to \Gamma(U, F)) \to \ker J_x^\infty D$$

is injective. •

Remark 1.6.5. If D has smooth coefficients, satisfies $D^*D = \nabla^*\nabla$ + lower order terms, and U is connected, then it has the strong unique continuation property at $x \in U$ [Cor56; Aro57]; see also [GL87; Kaz88] for stream-lined proofs using Almgren's frequency function. ♣

Proposition 1.6.6. *Assume Situation 1.3.1 (or Situation 1.4.1). Let $x \in M$. Let $U \subset M$ be an open neighborhood of x. Let $D\colon \Gamma(E) \to \Gamma(F)$ be a linear elliptic differential operator with smooth coefficients. Suppose that D and $D^\dagger$ possess the strong unique continuation property at x in U. If $J_x^\infty D$ satisfies the ∞–jet Petri condition, then D satisfies the $\mathfrak{V}$–equivariant (or G–equivariant) Petri condition in U.*

Proof. By Section 1.5, it suffices to consider Situation 1.3.1. Set $G \coloneqq \pi_1(M, x_0)$. Denote by $\pi\colon (\tilde{M}, \tilde{x}_0) \to (M, x_0)$ the universal covering map. Set $\tilde{U} \coloneqq \pi^{-1}(U)$. The upcoming arguments prove that

$$\varpi_{\pi^*D,\tilde{U}}\colon \ \ker \pi^*D \otimes \ker(\pi^*D)^\dagger \to \Gamma(\tilde{U}, \pi^*E \otimes \pi^*F^\dagger)$$

is injective. Let $\tilde{x} \in \pi^{-1}(x)$. Since π is a covering map,

$$J_x^\infty D = J_{\tilde{x}}^\infty \pi^*D \quad \text{and} \quad J_x^\infty D^\dagger = J_{\tilde{x}}^\infty \pi^*D^\dagger.$$

Therefore, there is a commutative diagram

$$\begin{tikzcd}
\ker \pi^*D \otimes \ker \pi^*D^\dagger \arrow[r, "\varpi_{D,U}"] \arrow[d] & \Gamma(\tilde{U}, \pi^*E \otimes \pi^*F^\dagger) \arrow[d] \\
\ker J_x^\infty D \otimes \ker J_x^\infty D^\dagger \arrow[r, "\varpi_{J_x^\infty D}"] & J_x^\infty(E \otimes F^\dagger).
\end{tikzcd}$$

Since D and $D^\dagger$ have the strong unique continuation property at $x \in U$, π^*D and $\pi^*D^\dagger$ have the strong unique continuation property at $\tilde{x} \in \tilde{U}$. Consequently, the left vertical map is injective. Therefore, since $\varpi_{J_x^\infty D}$ is injective, so is $\varpi_{\pi^*D,\tilde{U}}$.

Every $s \in \Gamma(E \otimes \underline{V}_\alpha)$ can be regarded as an element $\tilde{s} \in (\Gamma(\pi^*E) \otimes V_\alpha)^G$. This establishes an isomorphism $\ker D_p^{\underline{V}_\alpha} \cong (\ker \pi^*D_p \otimes V_\alpha)^G$ (and similarly for $D^\dagger$). Consider the commutative diagram

$$\begin{tikzcd}
\bigoplus_{\alpha=1}^m \ker D_p^{\underline{V}_\alpha} \otimes_{\mathbf{K}_\alpha^{\mathrm{op}}} \ker D_p^{\dagger,\underline{V}_\alpha^*} \arrow[r, "\cong"] \arrow[dd, "\varpi_{D,U}^{\mathfrak{V}}"'] & \bigoplus_{\alpha=1}^m (\ker \pi^*D_p \otimes V_\alpha)^G \otimes_{\mathbf{K}_\alpha^{\mathrm{op}}} (\ker \pi^*D_p^\dagger \otimes V_\alpha^*)^G \arrow[d, "\mathrm{tr}"] \\
 & (\ker \pi^*D_p \otimes \ker \pi^*D_p^\dagger)^G \arrow[d, "\varpi_{\pi^*D,\tilde{U}}"] \\
\Gamma(U, E \otimes F^\dagger) \arrow[r, "\cong"'] & \Gamma(\tilde{U}, \pi^*E \otimes \pi^*F^\dagger).
\end{tikzcd}$$

Here tr is the sum of the maps induced by the trace maps $V_\alpha \otimes V_\alpha^* \to \mathbf{R}$. It is a consequence of Proposition 1.3.14 that the map tr is injective. Therefore, since $\varpi_{\pi^*D,\tilde{U}}$ is injective, so is $\varpi_{D,U}^{\mathfrak{V}}$. ■

The failure of the ∞–jet Petri condition manifests itself at the level of symbols as follows.

Definition 1.6.7. Let $k \in \mathbf{N}_0$. A **symbol of order** k is an element $\sigma \in S^k T_x M \otimes \mathrm{Hom}(E_x, F_x)$. Since every $v \in T_x M$ defines a derivation ∂_v on the polynomial algebra $S^\bullet T_x^* M$, every symbol σ defines a **formal differential operator**

$$\hat{\sigma}\colon\ S^\bullet T_x^* M \otimes E_x \to S^\bullet T_x^* M \otimes F_x.$$

The **adjoint symbol** $\sigma^\dagger \in S^k T_x M \otimes \mathrm{Hom}(F_x^\dagger, E_x^\dagger)$ is $(-1)^k$–times the image of σ under the the map induced by the canonical isomorphism $\mathrm{Hom}(E_x, F_x) \cong \mathrm{Hom}(F_x^\dagger, E_x^\dagger)$. •

Remark 1.6.8. Here is an explicit description of the above provided a basis $(\partial_1, \dots, \partial_n)$ of $T_x M$ has been chosen. Denote the dual basis of $T_x^* M$ by $(x_1, \dots, x_n)$ and set $\partial_i x_j := x_i(\partial_i)$ $(= \delta_{ij})$. This exhibits $S^\bullet T_x^* M$ as the polynomial ring $\mathbf{R}[x_1, \dots, x_n]$. Evidently, ∂_i acts on $\mathbf{R}[x_1, \dots, x_n]$ by differentiation. If the symbol σ is

$$\sigma = \sum_{|\alpha|=k} \partial^\alpha \otimes a_\alpha$$

with the sum taken over all multi-indices $\alpha \in \mathbf{N}_0^n$ of length $|\alpha| = k$, then

$$\hat{\sigma} = \sum_{|\alpha|=k} a_\alpha \cdot \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}.$$ ♣

The symbol of a linear differential operator $D\colon \Gamma(E) \to \Gamma(F)$ of order k is a section $\sigma(D) \in \Gamma(S^k TM \otimes \mathrm{Hom}(E, F))$. Its value $\sigma_x(D)$ at $x \in M$ is a symbol in the above sense and depends only on $J_x^0 D$. Furthermore, $\sigma_x(D^\dagger) = \sigma_x(D)^\dagger$.

Definition 1.6.9. The **polynomial Petri** map $\hat{\varpi}\colon (S^\bullet T_x^* M \otimes E_x) \otimes (S^\bullet T_x^* M \otimes F_x^\dagger) \to S^\bullet T_x^* M \otimes E_x \otimes F_x^\dagger$ is defined by

$$\hat{\varpi}((p \otimes e) \otimes (q \otimes f)) := (p \cdot q) \otimes e \otimes f.$$

A symbol $\sigma \in S^k T_x M \otimes \mathrm{Hom}(E_x, F_x)$ satisfies the **polynomial Petri condition** if the map

$$\hat{\varpi}_\sigma\colon\ \ker \hat{\sigma} \otimes \ker \hat{\sigma}^\dagger \to S^\bullet T_x^* M \otimes E_x \otimes F_x^\dagger$$

induced by the polynomial Petri map is injective. •

Proposition 1.6.10. *If $J_x^\infty D$ fails to satisfy the ∞–jet Petri condition, then $\sigma_x(D)$ fails to satisfy the polynomial Petri condition.*

The proof of this result and the upcoming discussion require the following algebraic definitions, constructions, and facts:

(1) Let V be a vector space equipped with a filtration $\mathscr{F}_\bullet V$. The **order of** $\mathscr{F}_\bullet V$ is the map $\mathrm{ord}\colon V \to \mathbf{Z} \cup \{\infty, -\infty\}$ defined by

$$\mathrm{ord}(v) := \sup\{j \in \mathbf{Z} : v \in \mathscr{F}_j V\}.$$

$\mathscr{F}_\bullet V$ is called **exhaustive** if $\mathrm{ord}^{-1}(-\infty) = \varnothing$ or, equivalently, $\bigcup_{j \in \mathbf{Z}} \mathscr{F}_j V = V$. $\mathscr{F}_\bullet V$ is **separated** if $\mathrm{ord}^{-1}(\infty) = 0$ or, equivalently, $\bigcap_{j \in \mathbf{Z}} \mathscr{F}_j V = 0$.

(2) The **associated graded vector space of $\mathscr{F}_\bullet V$** is

$$\operatorname{gr} V \coloneqq \bigoplus_{j\in\mathbf{Z}} \operatorname{gr}_j V \quad \text{with} \quad \operatorname{gr}_j V \coloneqq \mathscr{F}_j V/\mathscr{F}_{j+1}V.$$

Define $[\cdot]\colon\ \operatorname{ord}^{-1}(\mathbf{Z}) \to \operatorname{gr} V$ by

$$[v] \coloneqq v + \mathscr{F}_{j+1}V \in \operatorname{gr}_j V \quad \text{with} \quad j \coloneqq \operatorname{ord}(v).$$

This is map is not linear and not even continuous (except for a few corner cases). It is appropriate to regard $[v]$ as the leading order term of v.

(3) Let W be a further vector space equipped with a filtration $\mathscr{F}_\bullet W$. A linear map $f\colon\ V \to W$ is **of order** $k \in \mathbf{Z}$ if $f(\mathscr{F}_j V) \subset \mathscr{F}_{j+k}W$ for every $j \in \mathbf{Z}$ but the same does not hold for $k+1$ instead of k. If this is the case, then f induces a linear map

$$\operatorname{gr} f\colon\ \operatorname{gr} V \to \operatorname{gr} W$$

of degree k. There is a canonical inclusion $\operatorname{gr}\ker f \hookrightarrow \ker\operatorname{gr} f$ and and a canonical projection $\operatorname{coker}\operatorname{gr} f \twoheadrightarrow \operatorname{gr}\operatorname{coker} f$. These maps are typically not isomorphisms.

(4) The tensor product $V \otimes W$ inherits the **tensor product filtration** defined by

$$\mathscr{F}_j(V\otimes W) \coloneqq \sum_{j_1+j_2=j} \mathscr{F}_{j_1}V \otimes \mathscr{F}_{j_2}W.$$

There is a canonical graded isomorphism

$$\operatorname{gr}(V\otimes W) \cong \operatorname{gr} V \otimes \operatorname{gr} W. \tag{1.6.11}$$

If $\mathscr{F}_\bullet V$ and $\mathscr{F}_\bullet W$ are both separated (exhaustive), then so is $\mathscr{F}_\bullet(V\otimes W)$.

Let $\ell \in \mathbf{N}_0 \cup \{\infty\}$. The vanishing order filtration on $\mathscr{E}_x$ descends to a filtration on $J_x^\ell E$. Taylor expansion defines an isomorphism

$$T_x^\ell\colon\ \operatorname{gr} J_x^\ell E \to \bigoplus_{j=0}^{\ell} S^j T_x^* M \otimes E_x; \tag{1.6.12}$$

in particular:

$$\dim J_x^\ell E = r \cdot \binom{n+\ell}{n} \tag{1.6.13}$$

(and similarly for $E^\dagger$, F, and $F^\dagger$ instead of E). If $D\colon\ \Gamma(E) \to \Gamma(F)$ is a linear differential operator and $\sigma_x(D)$ denotes its symbol at x, then

$$T_x^\infty \circ \operatorname{gr} J_x^\infty D = \hat\sigma_x(D) \circ T_x^\infty.$$

Furthermore,

$$T_x^\infty \circ \operatorname{gr} J_x^\infty \varpi = \hat\varpi \circ (T_x^\infty \otimes T_x^\infty).$$

This implies corresponding identities for $\ell \in \mathbf{N}_0$ instead of ∞ provided $\hat\sigma_x(D)$ and $\hat\varpi$ are appropriately truncated.

Proof of Proposition 1.6.10. The vanshing order filtration on $J_x^\infty E$ and $J_x^\infty F$ is exhaustive and separated. Therefore, if $B \in \ker \varpi_{J_x^\infty D}$ is non-zero, then

$$[B] \in (\ker \operatorname{gr} J_x^\infty D \otimes \ker \operatorname{gr} J_x^\infty D^\dagger) \cap \ker \operatorname{gr} J_x^\infty \varpi$$

is defined and non-zero. By the preceding discussion, T_x^∞ induces an isomorphism

$$(\ker \operatorname{gr} J_x^\infty D \otimes \ker \operatorname{gr} J_x^\infty D^\dagger) \cap \ker \operatorname{gr} J_x^\infty \varpi \cong \ker \hat{\varpi}_\sigma \quad \text{with} \quad \sigma = \sigma_x(D). \qquad \blacksquare$$

Proposition 1.6.10 is probably not terribly useful for establishing the ∞–jet Petri condition. The polynomial Petri condition fails for real Cauchy–Riemann operators (see [Wen19b, Example 5.5] and Proposition 2.5.4) and we suspect that it typically fails. However, this is no reason to despair. It can be shown that every $\hat{B} \in \ker \hat{\sigma}_x(D) \otimes \ker \hat{\sigma}_x(D^\dagger)$ admits some (but not a unique) lift to an element $B \in \ker J_x^\infty D \otimes \ker J_x^\infty D^\dagger$. However, if $\hat{B} \in \ker \hat{\varpi}$, then this does not imply that $B \in \ker J_x^\infty \varpi$. In fact, it is reasonable to expect that typically the higher order terms will prevent the vanishing of $J_x^\infty \varpi(B)$. The upcoming theorem shows that this heuristic is valid assuming an algebraic hypothesis on symbol level.

Definition 1.6.14. Let $k, \ell \in \mathbf{N}_0$. A family of linear elliptic differential operators $(D_p)_{p \in \mathscr{P}}$ of order k with smooth coefficients is **ℓ–jet flexible at x and $p_\star \in \mathscr{P}$** if for every $A \in J_x^\ell \operatorname{Hom}(E, F)$ there is a $\hat{p} \in T_{p_\star}\mathscr{P}$ such that

$$\mathrm{d}_{p_\star} J_x^\ell D(\hat{p}) s = As$$

for every $s \in J_x^{k+\ell} E$. •

Definition 1.6.15. Let $k \in \mathbf{N}_0$. A symbol $\sigma \in S^k T_x M \otimes \operatorname{Hom}(E_x, F_x)$ satisfies **Wendl's condition** if there are $c_0 \colon \mathbf{N}_0 \times \mathbf{N} \to (0, \infty)$ and $\ell_0 \colon \mathbf{N}_0 \times \mathbf{N} \to \mathbf{N}_0$ such that for every every homogeneous $B \in \ker \hat{\varpi}_\sigma$ the following hold: there are right-inverses $\hat{R}$ and $\hat{R}^\dagger$ of $\hat{\sigma}$ and $\hat{\sigma}^\dagger$ such that the linear map

$$\hat{\mathbf{L}}_{\sigma,B} \colon S^\bullet T_x^* M \otimes \operatorname{Hom}(E_x, F_x) \to S^\bullet T_x^* M \otimes E_x \otimes F_x^\dagger$$

defined by

$$\hat{\mathbf{L}}_{\sigma,B}(A) \coloneqq \hat{\varpi}\big((\hat{R}A \otimes \mathbf{1} + \mathbf{1} \otimes \hat{R}^\dagger A^\dagger)B\big)$$

satisfies

$$\operatorname{rk} \hat{\mathbf{L}}_{\sigma,B}^{\leqslant \ell} \geqslant c_0(d, \rho)\ell^n$$

for every $\ell \geqslant \ell_0(d, \rho)$ with $d \coloneqq \deg B$ and $\rho \coloneqq \operatorname{rk} B$. Here

$$\hat{\mathbf{L}}_{\sigma,B}^{\leqslant \ell} \colon \bigoplus_{j=0}^{\ell} S^j T_x^* M \otimes \operatorname{Hom}(E_x, F_x) \to \bigoplus_{j=0}^{k+\ell} S^j T_x^* M \otimes E_x \otimes F_x^\dagger$$

denotes the truncation of $\hat{\mathbf{L}}_{\sigma,B}$. •

Remark 1.6.16. The reader is by no means expected to understand the significance of Wendl's condition at this point. The following remarks might help clarify the definition:

(1) Proposition 1.6.20 proves that $\hat{\sigma}$ and $\hat{\sigma}^\dagger$ have right-inverses provided σ is elliptic.

(2) The maps $\hat{\mathbf{L}}^{\leqslant \ell}_{\sigma,B}$ play a crucial role in the proof of Theorem 1.6.17. Their ranks provide lower bounds for the ranks of certain map between jet spaces tied to the failure of the ∞–jet Petri condition.

(3) The dimension of the codomain of $\hat{\mathbf{L}}^{\leqslant \ell}_{\sigma,B}$ grows like ℓ^n; therefore, $\operatorname{rk} \hat{\mathbf{L}}^{\leqslant \ell}_{\sigma,B}$ is assumed to have maximal growth rate.

(4) Unfortunately, it appears not to be easy to verify whether a given symbol σ satisfies Wendl's condition or not. In fact, even determining $\ker \hat{\varpi}_\sigma$ is a non-trivial task. Theorem 2.5.3 proves that the symbol σ of a real Cauchy–Riemann operator satisfies Wendl's condition. As far as we know, it is possible that every elliptic symbol satisfies Wendl's condition. ♣

Theorem 1.6.17 (Wendl [Wen19b, Section 5.2]). *Let $(D_p)_{p\in\mathscr{P}}$ be a family of linear elliptic differential operators with smooth coefficients of order k. Set*

$$\mathscr{R} \coloneqq \{p \in \mathscr{P} : J^\infty_x D \text{ fails to satisfy the } \infty\text{–jet Petri condition}\}.$$

Let $p_\star \in \mathscr{P}$. If

(1) *$(D_p)_{p\in\mathscr{P}}$ is ℓ–jet flexible at x and $p_\star \in \mathscr{P}$ for every $\ell \in \mathbf{N}_0$, and*

(2) *the symbol $\sigma_x(D_{p_\star})$ satisfies Wendl's condition,*

then for every $c \in \mathbf{N}_0$ there is an open neighborhood $\mathscr{U}$ of $p_\star \in \mathscr{P}$ such that $\mathscr{R}\cap\mathscr{U}$ has codimension at least c.[9]

The remainder of this section is devoted to the proof of Theorem 1.6.17. The following observation decomposes $\mathscr{R}$ into pieces whose codimensions can be estimated using the hypotheses of the theorem.

Proposition 1.6.18. *For $d \in \mathbf{N}_0$ and $\rho \in \mathbf{N}$ set*

$$\mathscr{R}^\ell_{d,\rho} \coloneqq \left\{ p \in \mathscr{P} : \begin{array}{l} \text{there is a } B \in (\ker J^\ell_x D_p \otimes \ker J^\ell_x D_p^\dagger) \cap \ker J^{k+\ell}_x \varpi \\ \text{with } \operatorname{ord}(B) \leqslant d \text{ and } \operatorname{rk} B = \rho \end{array} \right\}.$$

The set $\mathscr{R}$ satisfies

$$\mathscr{R} \subset \bigcup_{\substack{d\in\mathbf{N}_0\\ \rho\in\mathbf{N}\\ \ell_0\in\mathbf{N}_0}} \bigcap_{\ell\geqslant\ell_0} \mathscr{R}^\ell_{d,\rho}.$$

The proof relies on the following fact.

Proposition 1.6.19. *Let V be a vector space and equipped with a filtration $\mathscr{F}_\bullet V$. Set*

$$Q_\ell \coloneqq V/\mathscr{F}_\ell V \quad \text{and} \quad Q \coloneqq \varprojlim Q_\ell.$$

If $R \subset Q$ is a finite dimensional subspace, then there is an $\ell_0 \in \mathbf{N}_0$ such that for every $\ell \geqslant \ell_0$ the restriction of the composition $R \to Q \to Q_\ell$ is injective.

[9] Definition 1.B.1 defines what it means for a subset of a Banach manifold to have codimension at least c.

Proof. $K_\ell := \ker(R \to Q \to Q_\ell)$ is a decreasing sequence of finite dimensional vector spaces with $\varprojlim K_\ell = 0$. Therefore, $K_\ell = 0$ for $\ell \gg 1$. ■

Proof of Proposition 1.6.18. If $p \in \mathscr{R}$, then there exists a non-zero $B \in \ker \varpi_{J^\infty_x D_p}$. Set $d := \operatorname{ord}(B)$, $\rho := \operatorname{rk} B$, and write B as

$$B = \sum_{i=1}^{\rho} s_i \otimes t_i.$$

with $s_1, \dots, s_\rho$ and $t_1, \dots, t_\rho$ linearly independent. Since

$$J^\infty_x E = \varprojlim J^\ell_x E$$

and by Proposition 1.6.19, there is an $\ell_0 \in \mathbf{N}_0$ such that for every $\ell \geqslant \ell_0$ the $(k+\ell)$–jets $\tilde{s}_1, \dots, \tilde{s}_\rho \in J^{k+\ell}_x E$ and $\tilde{t}_1, \dots, \tilde{t}_\rho \in J^{k+\ell}_x F^\dagger$ are linearly independent. By construction,

$$\tilde{B} := \sum_{i=1}^{\rho} \tilde{s}_i \otimes \tilde{t}_i \in \ker \varpi_{J^\ell_x D}$$

satisfies

$$\operatorname{ord}(\tilde{B}) = d \quad \text{and} \quad \operatorname{rk} \tilde{B} = \rho.$$

Therefore, $p \in \mathscr{R}^\ell_{d,\rho}$ for every $\ell \geqslant \ell_0$. ■

To estimate the codimension of $\mathscr{R}^\ell_{d,\rho}$ we require the following. Recall that $\dim M = n$ and $\operatorname{rk} E = \operatorname{rk} F = r$.

Proposition 1.6.20. *Let $J^\infty_x D$ be the ∞–jet of an elliptic differential operator D of order k. The following hold:*

(1) *The formal differential operator $\hat{\sigma}_x(D)$ is surjective.*

(2) *For every $\ell \in \mathbf{N}_0 \cup \{\infty\}$ the ℓ–jet $J^\ell_x D$ is surjective.*

(3) *For every $\ell \in \mathbf{N}_0$*

$$\dim \ker J^\ell_x D = r \cdot \left[\binom{n+k+\ell}{n} - \binom{n+\ell}{n} \right].$$

Proof. Since D is elliptic, the restriction $\hat{\sigma}^k_x(D) \colon S^k T^*_x M \otimes E_x \to F_x$ is surjective. Choose a basis $(\xi_1, \dots, \xi_n)$ of $T^*_x M$. For a multi-index $\alpha \in \mathbf{N}^n_0$ set $\xi^\alpha := \prod_{i=1}^n \xi_i^{\alpha_i}$. A moment's thought shows that

$$\hat{\sigma}_x(D)(\xi_1^k \xi^\alpha \otimes e) = \binom{k+\alpha_1}{\alpha_1} \xi^\alpha \otimes \hat{\sigma}_x(D)(\xi_1^k \otimes e) + R$$

with R denoting a sum of tensors of the form $\xi^\beta \otimes w$ with $\beta_1 > \alpha_1$. Therefore, the image of $\hat{\sigma}_x(D)$ contains every tensor product of the form $\xi_1^m \otimes f$. Descending induction on α_1 starting at m proves that the image of $\hat{\sigma}_x(D)$ contains every tensor product of the form $\xi^\alpha \otimes w$ with $|\alpha| = m$. This proves (1).

Since $\operatorname{coker} \operatorname{gr} J^\ell_x D \twoheadrightarrow \operatorname{gr} \operatorname{coker} J^\ell_x D$, (1) implies (2).

Finally, (2) implies $\dim \ker J^\ell_x D = \dim J^{k+\ell}_x E - \dim J^\ell_x F$. Therefore, (3) follows from (1.6.13). ■

Proof of Theorem 1.6.17. Let $d \in \mathbf{N}_0$, $\rho \in \mathbf{N}$, and $\ell \geqslant \ell_0(d,\rho)$. By Proposition 1.6.20,

$$\mathscr{K}^\ell := \{(p,s) \in \mathscr{P} \times J_x^{k+\ell}E : s \in \ker J_x^\ell D_p\} \quad \text{and}$$
$$\mathscr{C}^\ell := \{(p,t) \in \mathscr{P} \times J_x^{k+\ell}F^\dagger : t \in \ker J_x^\ell D_p^\dagger\}$$

are vector bundles over $\mathscr{P}$ of rank

$$\operatorname{rk}\mathscr{K}^\ell = \operatorname{rk}\mathscr{C}^\ell = r \cdot \left[\binom{n+k+\ell}{n} - \binom{n+\ell}{n}\right].$$

Therefore,

$$\mathscr{T}_{d,\rho}^\ell := \{(p,B) \in \mathscr{K}^\ell \otimes \mathscr{C}^\ell : \operatorname{ord}(B) \leqslant d \text{ and } \operatorname{rk} B = \rho\}$$

is a fiber bundle over $\mathscr{P}$ with fibers of dimension

$$2\rho r \cdot \left[\binom{n+k+\ell}{n} - \binom{n+\ell}{n}\right] - \rho^2 \leqslant c(n,k)\rho\ell^{n-1}.$$

Denote by $\pi\colon \mathscr{T}_{d,\rho}^\ell \to \mathscr{P}$ the projection map. By construction,

$$\mathscr{R}_{d,\rho}^\ell = \pi((J_x^{k+\ell}\varpi \circ \mathrm{pr}_2)^{-1}(0)).$$

The upcoming discussion proves that for every $(p_\star, B) \in (J_x^{k+\ell}\varpi \circ \mathrm{pr}_2)^{-1}(0)$

$$\operatorname{rk} \mathrm{d}_{(p_\star,B)}(J_x^{k+\ell}\varpi \circ \mathrm{pr}_2) \geqslant c_0(d,\rho)\ell^n.$$

Therefore, there is an open neighborhood $\mathscr{U}$ of $p_\star \in \mathscr{P}$ such that for every $(p,B) \in (J_x^{k+\ell}\varpi \circ \mathrm{pr}_2)^{-1}(0)$ with $p \in \mathscr{U}$ the analogous condition hold. Consequently, by Proposition 1.B.2, $\mathscr{R}_{d,\rho}^\ell \cap \mathscr{U}$ has codimension at least

$$(c_0(d,\rho)\ell - c(n,k)\rho)\ell^n.$$

This immediately implies the theorem.

Let $(p_\star, B) \in (J_x^{k+\ell}\varpi \circ \mathrm{pr}_2)^{-1}(0)$. Set $\sigma := \sigma_x(D_{p_\star})$. Denote by $\hat{R}$ and $\hat{R}^\dagger$ the right-inverses of $\hat{\sigma}$ and $\hat{\sigma}^\dagger$ from Definition 1.6.15. Denote by R and $R^\dagger$ right-inverses of $J_x^{k+\ell}D_{p_\star}$ and $J_x^{k+\ell}D_{p_\star}^\dagger$ such that $\operatorname{gr} R$ and $\operatorname{gr} R^\dagger$ correspond to the truncations of $\hat{R}$ and $\hat{R}^\dagger$ with respect to (1.6.12). Define $\mathbf{L}_{p_\star,B}\colon J_x^\ell \operatorname{Hom}(E_x,F_x) \to J_x^{k+\ell}(E_x \otimes F_x^\dagger)$ by

$$\mathbf{L}_{p_\star,B}(A) := J_x^{k+\ell}\varpi\big((RA \otimes \mathbf{1} + \mathbf{1} \otimes R^\dagger A^\dagger)B\big)$$

Since $(D_p)_{p\in\mathscr{P}}$ is ℓ–jet flexible at x and $p_\star \in \mathscr{P}$, for every $A \in J_x^\ell \operatorname{Hom}(E_x,F_x)$ there is a $\hat{p} \in T_{p_\star}\mathscr{P}$ such that

$$\mathrm{d}_{p_\star} J_x^\ell D(\hat{p})s = As$$

for every $s \in J_x^{k+\ell}E$. If this identity holds, then

$$(\hat{p}, (RA \otimes \mathbf{1} + \mathbf{1} \otimes R^\dagger A^\dagger)B) \in T_{(p_\star,B)}\mathscr{T}_{d,\rho}^\ell.$$

Therefore,

$$\operatorname{rk} \mathrm{d}_{(p,B)}(J_x^{k+\ell}\varpi \circ \mathrm{pr}_2) \geqslant \operatorname{rk} \mathbf{L}_{p_\star,B}.$$

Since $\operatorname{gr}\ker \mathbf{L}_{p_\star,B} \hookrightarrow \ker\operatorname{gr}\mathbf{L}_{p_\star,B}$,

$$\operatorname{rk}\mathbf{L}_{p_\star,B} = r\cdot\binom{n+\ell}{n} - \dim\operatorname{gr}\ker\mathbf{L}_{p_\star,B} \geqslant r\cdot\binom{n+\ell}{n} - \dim\ker\operatorname{gr}\mathbf{L}_{p_\star,B} = \operatorname{rk}\operatorname{gr}\mathbf{L}_{p_\star,B}.$$

The isomorphism (1.6.12) identifies $[B]$ with $\hat{B} \in \ker\hat{\varpi}_\sigma$, which is homogeneous of degree d. Furthermore, it identifies $\operatorname{gr}\mathbf{L}_{p_\star,B}$ with $\hat{\mathbf{L}}^{\leqslant\ell}_{\sigma,\hat{B}}$. Therefore,

$$\operatorname{rk}\mathrm{d}_{(p,B)}(J^{k+\ell}_x\varpi\circ\mathrm{pr}_2) \geqslant \operatorname{rk}\hat{\mathbf{L}}^{\leqslant\ell}_{\sigma,\hat{B}} \geqslant c_0(d,\rho)\ell^n.$$

This finishes the proof. ■

1.A Self-adjoint operators

The purpose of this section is to summarize how the results developed so far need to be modified to become useful for families of formally self-adjoint linear elliptic differential operators. These are relevant for many geometric applications (although not for Part 2.) A particularly interesting application would be to understand the generic multiple cover phenomena for associative submanifolds in G_2–manifolds: the deformation theory of the latter are controlled by twisted Dirac operators.

Definition 1.A.1. Let $k \in \mathbf{N}_0$. A **family of formally self-adjoint linear elliptic differential operators** of order k consists of a Banach manifold $\mathscr{P}$ and a smooth map

$$D\colon \mathscr{P} \to \mathscr{F}(W^{k,2}\Gamma(E), L^2\Gamma(E))$$

such that for every $p \in \mathscr{P}$ the operator $D_p \coloneqq D(p)$ is a formally self-adjoint linear elliptic differential operator of order k. •

Throughout this section, assume Situation 1.3.1 and keep the following in mind:

(1) The algebras $\mathbf{K}_\alpha$ carry an anti-involution $\lambda \mapsto \lambda^*$ and an inner product $\langle\lambda,\mu\rangle \coloneqq \operatorname{tr}(\mu^*\lambda)$. (These correspond to the standard conjugation and inner products on $\mathbf{R}$ $\mathbf{C}$, and $\mathbf{H}$.) The Euclidean metric on $\underline{V}_\alpha$ is $\mathbf{K}_\alpha$–sesquilinear.

(2) Let W be a left $\mathbf{K}_\alpha$–module equipped with a $\mathbf{K}_\alpha$–sesquilinear inner product. Denote by $\operatorname{Sym}_{\mathbf{K}_\alpha}(W)$ the space of self-adjoint $\mathbf{K}_\alpha$–linear maps. W is a right $\mathbf{K}_\alpha$–module with $v\cdot\lambda \coloneqq \lambda^*\cdot v$. Therefore, one can form the tensor product $W\otimes_{\mathbf{K}_\alpha} W$ and the symmetric tensor power $S^2_{\mathbf{K}_\alpha}W$.

(3) Let D be a formally self-adjoint linear elliptic differential operator. The $\mathbf{K}_\alpha$–sesquilinear inner product on $\underline{V}_\alpha$ induces a canonical isomorphism

$$\ker D_p^{\underline{V}_\alpha} \cong \operatorname{coker} D_p^{\underline{V}_\alpha}.$$

Moreover, the map Λ_p^α defined in Theorem 1.3.5 takes values in

$$\operatorname{Sym}_{\mathbf{K}_\alpha}(\ker D_p^{\underline{V}_\alpha}).$$

Here is the analogue of the theory developed in Section 1.3. (It is left as an exercise for the reader to work out the analogues of Section 1.4 and Section 1.5.)

Definition 1.A.2. Let $(D_p)_{p\in\mathscr{P}}$ be a family of formally self-adjoint linear elliptic differential operators. For $d \in \mathbf{N}_0^m$ define the **$\mathfrak{V}$–equivariant self-adjoint Brill–Noether locus** $\mathscr{P}_d^{\mathfrak{V}}$ by

$$\mathscr{P}_d^{\mathfrak{V}} := \left\{p \in \mathscr{P} : \dim_{\mathbf{K}_i} \ker D_p^{\underline{V}_i} = d_i\right\}. \qquad \bullet$$

Theorem 1.A.3. *Let $(D_p)_{p\in\mathscr{P}}$ be a family of formally self-adjoint linear elliptic differential operators. For $p \in \mathscr{P}$ define $\Lambda_p^{\mathfrak{V}} \colon T_p\mathscr{P} \to \bigoplus_{\alpha=1}^m \mathrm{Sym}_{\mathbf{K}_\alpha}(\ker D_p^{\underline{V}_\alpha})$ by*

$$\Lambda_p^{\mathfrak{V}}(\hat{p}) := \bigoplus_{\alpha=1}^m \Lambda_p^\alpha(\hat{p}) \quad \textit{and} \quad \Lambda_p^\alpha(\hat{p})s := \mathrm{d}_p D^{\underline{V}_\alpha}(\hat{p})s \mod \operatorname{im} D_p^{\underline{V}_\alpha}.$$

Let $d \in \mathbf{N}_0^m$ and $p \in \mathscr{P}_d^{\mathfrak{V}}$. If $\Lambda_p^{\mathfrak{V}}$ is is surjective, then the following hold:

1. *There is an open neighborhood $\mathscr{U}$ of $p \in \mathscr{P}$ such that $\mathscr{P}_d^{\mathfrak{V}} \cap \mathscr{U}$ is a submanifold of codimension*

$$\operatorname{codim}(\mathscr{P}_d^{\mathfrak{V}} \cap \mathscr{U}) = \sum_{\alpha=1}^m d_\alpha + k_\alpha \binom{d_\alpha}{2}.$$

2. *$\mathscr{P}_{\tilde{d}}^{\mathfrak{V}} \neq \varnothing$ for every $\tilde{d} \in \mathbf{N}_0^m$ with $\tilde{d} \leqslant d$.* ■

Proof. There is a straight-forward variation of Lemma 1.1.4 to self-adjoint Fredholm operators. This reduces the proof to the finite-dimensional situation. The latter is straightforward. The codimension formula follows from

$$\dim \mathrm{Sym}_{\mathbf{K}}(\mathbf{K}^d) = d + k\binom{d}{2}$$

with $k := \dim_{\mathbf{R}} \mathbf{K}$. ■

Definition 1.A.4. A family of linear elliptic differential operators $(D_p)_{p\in\mathscr{P}}$ is **$\mathfrak{V}$–equivariantly symmetrically flexible in U at $p_\star \in \mathscr{P}$** if for $A \in \Gamma_c(U, \mathrm{Sym}(E))$ there is a $\hat{p} \in T_{p_\star}\mathscr{P}$ such that

$$\mathrm{d}_{p_\star} D^{\underline{V}_\alpha}(\hat{p})s = (A \otimes \mathrm{id}_{\underline{V}_\alpha})s \mod \operatorname{im} D_{p_\star}^{\underline{V}_\alpha}$$

for every $\alpha = 1, \dots, m$ and $s \in \ker D_p^{\underline{V}_\alpha}$. •

Definition 1.A.5. The **$\mathfrak{V}$–equivariant symmetric Petri map**

$$\varsigma^{\mathfrak{V}} \colon \bigoplus_{\alpha=1}^m S^2_{\mathbf{K}_\alpha}\Gamma(E \otimes \underline{V}_\alpha) \to \Gamma(S^2E)$$

is defined by $\varsigma^{\mathfrak{V}} := \sum_{\alpha=1}^m \varsigma_\alpha$ with ς_α denoting the composition of the Petri map

$$\varsigma_\alpha \colon S^2_{\mathbf{K}_\alpha}\Gamma(E \otimes \underline{V}_\alpha) \to \Gamma(S^2E \otimes S^2_{\mathbf{K}_\alpha}\underline{V}_\alpha)$$

and the map induced by the inner product $\langle\cdot,\cdot\rangle\colon\ S^2_{\mathbf{K}_\alpha}\underline{V_\alpha}\to\underline{\mathbf{R}}$. Let $U\subset M$ be an open subset. A self-adjoint linear elliptic differential operator $D\colon\ \Gamma(E)\to\Gamma(E)$ satisfies the **$\mathfrak{V}$–equivariant symmetric Petri condition in** U if the map

$$\varsigma^{\mathfrak{V}}_{D,U}\colon\ \bigoplus_{\alpha=1}^{m} S^2_{\mathbf{K}_\alpha}\ker D_p^{\underline{V_\alpha}}\to L^1\Gamma(U,S^2E)$$

induced by the $\mathfrak{V}$–equivariant symmetric Petri map is injective. •

Proposition 1.A.6. *Let $(D_p)_{p\in\mathscr{P}}$ be a family of self-adjoint linear elliptic differential operators. Let $U\subset M$ be an open subset. If $(D_p)_{p\in\mathscr{P}}$ is $\mathfrak{V}$–equivariantly symmetrically flexible in U at $p_\star\in\mathscr{P}$ and $D_{p_\star}$ satisfies the $\mathfrak{V}$–equivariant symmetric Petri condition in U, then the map $\Lambda^{\mathfrak{V}}_{p_\star}$ defined in Theorem 1.A.3 is surjective.* ■

Remark 1.3.12 carries over mutatis mutandis; in particular, (1.3.13) it is still sharp for self-adjoint operators. Finally, these are the analogues of the results from Section 1.6.

Definition 1.A.7. The ∞–jet of a formally self-adjoint linear elliptic differential operator $J^\infty_x D\colon\ J^\infty_x E\to J^\infty_x E$ satisfies the **∞–jet symmetric Petri condition** if the map

$$\varpi_{J^\infty_x D}\colon\ \ker S^2 J^\infty_x D\to J^\infty_x S^2E$$

induced by the symmetric Petri map is injective. •

Proposition 1.A.8. *Assume Situation 1.3.1. Let $x\in M$. Let $U\subset M$ be an open neighborhood of x. Let $D\colon\ \Gamma(E)\to\Gamma(E)$ be a formally self-adjoint linear elliptic differential operator with smooth coefficients. Suppose that D possess the strong unique continuation property at x in U. If $J^\infty_x D$ satisfies the ∞–jet symmetric Petri condition, then D satisfies the $\mathfrak{V}$–equivariant symmetric Petri condition in U.*

Definition 1.A.9. The **polynomial symmetric Petri map** $\hat\varsigma\colon\ S^2(S^\bullet T^*_xM\otimes E_x)\to S^\bullet T^*_xM\otimes S^2E_x$ is defined as the restriction of the polynomial Petri map. A symmetric symbol $\sigma\in S^kT_xM\otimes\mathrm{Sym}(E_x)$ satisfies the **polynomial symmetric Petri condition** if the map

$$\hat\varsigma_\sigma\colon\ S^2\ker\hat\sigma\to S^\bullet T^*_xM\otimes S^2E_x$$

induced by the polynomial symmetric Petri map is injective. •

Proposition 1.A.10. *If $J^\infty_x D$ fails to satisfy the ∞–jet symmetric Petri condition, then $\sigma_x(D)$ fails to satisfy the polynomial symmetric Petri condition.* ■

Definition 1.A.11. Let $k,\ell\in\mathbf{N}_0$. A family of self-adjoint linear elliptic differential operators $(D_p)_{p\in\mathscr{P}}$ of order k is **ℓ–jet symmetrically flexible at x and $p_\star\in\mathscr{P}$** if for every $A\in J^\ell_x\,\mathrm{Sym}(E_x,F_x)$ there is a $\hat p\in T_{p_\star}\mathscr{P}$ such that

$$\mathrm{d}_{p_\star}J^\ell_x D(\hat p)s=As$$

for every $s\in J^{k+\ell}_x E$. •

Definition 1.A.12. Let $k \in \mathbf{N}_0$. A symbol $\sigma \in S^k T_x M \otimes \mathrm{Sym}(E_x)$ satisfies the **symmetric Wendl condition** if there are $c_0 \colon \mathbf{N}_0 \times \mathbf{N} \to (0,\infty)$ and $\ell_0 \colon \mathbf{N}_0 \times \mathbf{N} \to \mathbf{N}_0$ such that for every every homogeneous $B \in \ker \hat{\varsigma}_\sigma$ the following hold: there is a right-inverses $\hat{R}$ of $\hat{\sigma}$ such that the linear map

$$\hat{\mathbf{L}}_{\sigma,B} \colon\ S^\bullet T_x^* M \otimes \mathrm{Sym}(E_x) \to S^\bullet T_x^* M \otimes S^2 E_x$$

defined by

$$\hat{\mathbf{L}}_{\sigma,B}(A) \coloneqq \hat{\varpi}\big((\hat{R}A \otimes \mathbf{1} + \mathbf{1} \otimes \hat{R}A)B\big)$$

satisfies

$$\mathrm{rk}\, \hat{\mathbf{L}}_{\sigma,B}^{\leqslant \ell} \geqslant c_0(d,\rho)\ell^n$$

for every $\ell \geqslant \ell_0(d,\rho)$ with $d \coloneqq \deg B$ and $\rho \coloneqq \mathrm{rk}\, B$. Here

$$\hat{\mathbf{L}}_{\sigma,B}^{\leqslant \ell} \colon\ \bigoplus_{j=0}^{\ell} S^j T_x^* M \otimes \mathrm{Sym}(E_x) \to \bigoplus_{j=0}^{k+\ell} S^j T_x^* M \otimes S^2 E_x$$

denotes the truncation of $\hat{\mathbf{L}}_{\sigma,B}$. •

Remark 1.A.13. Verifying the symmetric Wendl condition appears to be the crucial issue in the geometric applications alluded to at the beginning of this section. In light of Theorem 2.5.3 it is tempting to conjecture that a typical twisted Dirac operator on a 3–manifold does satisfy this condition. ♣

Theorem 1.A.14 (Wendl [Wen19b, Section 5.2]). *Let $(D_p)_{p \in \mathscr{P}}$ be a family of formally self-adjoint linear elliptic differential operators with smooth coefficients of order k. Set*

$$\mathscr{R} \coloneqq \{p \in \mathscr{P} : J_x^\infty D \text{ fails to satisfy the } \infty\text{–jet symmetric Petri condition}\}.$$

Let $p_\star \in \mathscr{P}$. If

1. *$(D_p)_{p \in \mathscr{P}}$ is ℓ–jet symmetrically flexible at x and $p_\star \in \mathscr{P}$ for every $\ell \in \mathbf{N}_0$, and*
2. *the symbol $\sigma_x(D_{p_\star})$ satisfies the symmetric Wendl condition,*

then for every $c \in \mathbf{N}_0$ there is an open neighborhood $\mathscr{U}$ of $p_\star \in \mathscr{P}$ such that $\mathscr{R} \cap \mathscr{U}$ has codimension at least c.

Proof. The proof of Theorem 1.6.17 carries over with minor changes. The salient point is that

$$\mathscr{T}_{d,\rho}^{\ell} \coloneqq \big\{(p,B) \in S^2\mathscr{K}^\ell : \mathrm{ord}(B) \leqslant d \text{ and } \mathrm{rk}\, B = \rho\big\}.$$

is a fiber bundle over $\mathscr{P}$ of with fibers of dimension at most $c(n,k)\rho\ell^{n-1}$. ■

1.B Codimension in Banach manifolds

There are numerous possible definitions of the concept of codimension of a subset of a Banach manifold. The following is a minor variation of the definition from [BM15, Section 2.3] and particularly well-suited for the purposes of this article.

Definition 1.B.1. Let X be a Banach manifold and $c \in \mathbf{N}_0$. A subset $S \subset X$ has **codimension at least** c if there is a C^1 Banach manifold Z and a C^1 Fredholm map $\zeta\colon Z \to X$ such that

$$\sup_{z \in Z} \operatorname{index} \mathrm{d}_z\zeta \leqslant -c \quad \text{and} \quad S \subset \operatorname{im}\zeta.$$

The **codimension of** S is defined by

$$\operatorname{codim} S \coloneqq \sup\{c \in \mathbf{N}_0 : S \text{ is of codimension at least } c\} \in \mathbf{N}_0 \cup \{\infty\}. \qquad \bullet$$

The additivity of Fredholm indices implies the following.

Proposition 1.B.2. *Let X, Y be Banach manifolds. If $S \subset X$ and $f\colon X \to Y$ is a Fredholm map, then*

$$\operatorname{codim} f(S) \geqslant \operatorname{codim} S - \inf_{x \in X} \operatorname{index} \mathrm{d}_x f. \qquad \blacksquare$$

The codimension of a subset can be regarded as a measure of the non-genericity of its elements. In topology, one considers the following concepts.

Definition 1.B.3. Let X be a topological space and $S \subset X$. S is **meager** if it is contained in a countable union of closed subsets with empty interior. S is **comeager** if $X \backslash S$ is meager. $\bullet$

Recall from Footnote 2 that Banach manifolds are assumed to be Hausdorff, paracompact, and separable. The Baire category theorem asserts that a meager subset of a completely metrizable space (e.g., a Banach manifold) has empty interior or, equivalently, that every comeager subset of such a space is dense. In light of this, one often regards a meager subset as consisting of non-generic points and a comeager subset as consisting of generic points.

Proposition 1.B.4. *Let X be a Banach manifold and $S \subset X$. If $\operatorname{codim} S > 0$, then S is meager.*[10]

Proof. Let Z and ζ be as in Definition 1.B.1. Since $\operatorname{index} \mathrm{d}_z\zeta < 0$, by the Sard–Smale Theorem [Sma65, Theorem 1.3] $\operatorname{im}\zeta$ is meager; hence, so is S. $\blacksquare$

In practice, one often proves that a subset is meager by proving that it has positive codimension. The latter, however, yields more precise information.

Proposition 1.B.5. *Let M be a finite-dimensional manifold and let X be a Banach manifold. For every $S \subset X$ and $k \in \mathbf{N}$ the following hold:*

[10] The following stronger statement, which will not be used in this article, follows from Sard's theory of cotypes [Sar69]: S is contained in the countable union of closed subsets, none of which contains a submanifold of codimension $\operatorname{codim} S - 1$. In particular: since such closed subsets have empty interior, this condition implies that S is meager.

(1) *The subset* $\widetilde{S} \subset C^k(M,X)$ *consisting of those* f *such that* $f^{-1}(S) \neq \varnothing$ *satisfies*

$$\operatorname{codim} \widetilde{S} \geqslant \operatorname{codim} S - \dim M.$$

(2) *The subset consisting of those* $f \in C^k(M,X)$ *for which* $\operatorname{codim} f^{-1}(S) \geqslant \operatorname{codim} S$ *is comeager.*

Proof. Suppose that S has codimension at least c and let Z and ζ be as in Definition 1.B.1. Set $F := M \times C^k(M,X)$. The evaluation map $\mathrm{ev}\colon F \to X$ is a C^k submersion. Therefore,

$$\mathrm{ev}^* Z := \{(x,f;z) \in F \times Z : \mathrm{ev}(x,f) = \zeta(z)\}$$

is a C^k Banach manifold and the map $\mathrm{pr}_1\colon \mathrm{ev}^*Z \to F$ is a Fredholm map of index at most $-c$. The projection map $\mathrm{pr}_2\colon F \to C^k(M,X)$ is a Fredholm map of index $\dim M$.

To prove (1), observe that $\mathrm{ev}^{-1}(S) \subset \operatorname{im} \mathrm{pr}_1$. Therefore, $\operatorname{codim} \mathrm{ev}^{-1}(S) \geqslant c$; hence, by Proposition 1.B.2, $\widetilde{S} = \mathrm{pr}_2(\mathrm{ev}^{-1}(S))$ has codimension at least $c - \dim M$.

If $f \in C^k(M,X)$ is a regular value of $\mathrm{pr}_2 \circ \mathrm{pr}_1\colon \mathrm{ev}^*Z \to C^k(M,X)$, then $(\mathrm{pr}_2 \circ \mathrm{pr}_1)^{-1}(f)$ is a C^k submanifold of ev^*Z of dimension at most $\dim M - c$. Therefore, its projection to M has codimension at least c. A moment's thought shows that this projection is $f^{-1}(\operatorname{im}\zeta)$; hence, it contains $f^{-1}(S)$. Therefore, $\operatorname{codim} f^{-1}(S) \geqslant \operatorname{codim} S$. By the Sard–Smale Theorem, the set of regular values of $\mathrm{pr}_2 \circ \mathrm{pr}_1$ is comeager. This implies (2). ■

Part 2

Application to super-rigidity

2.1 Bryan and Pandharipande's super-rigidity conjecture

The notion of super-rigidity for holomorphic maps was first introduced in algebraic geometry by Bryan and Pandharipande [BP01, Section 1.2]. The purpose of this section is to recall the corresponding notion in symplectic geometry as defined by Eftekhary [Eft16, Section 1] and Wendl [Wen19b, Section 2.1].

Definition 2.1.1. Let (M,J) be an almost complex manifold. A J**–holomorphic map** $u\colon (\Sigma,j) \to (M,J)$ is a pair consisting of a closed, connected Riemann surface (Σ,j) and a smooth map $u\colon \Sigma \to M$ satisfying the non-linear Cauchy–Riemann equation

$$\bar\partial_J(u,j) := \frac{1}{2}(\mathrm{d}u + J(u) \circ \mathrm{d}u \circ j) = 0. \tag{2.1.2}$$

Let $u\colon (\Sigma,j) \to (M,J)$ be a J–holomorphic map. Let $\phi \in \mathrm{Diff}(\Sigma)$ be a diffeomorphism. The **reparametrization** of u by ϕ is the J–holomorphic map $u \circ \phi^{-1}\colon (\Sigma, \phi_* j) \to (M,J)$.

If $\pi\colon (\tilde\Sigma, \tilde j) \to (\Sigma,j)$ is a holomorphic map of degree $\deg(\pi) \geqslant 2$, then the composition $u \circ \pi\colon (\tilde\Sigma,\tilde j) \to (M,J)$ is said to be a **multiple cover of** u. A J–holomorphic map is **simple** if it is not constant and not a multiple cover. •

Super-rigidity is a condition on the infinitesimal deformation theory of the images of J–holomorphic maps (up to reparametrization). To give the precise definition, let us recall the salient parts of this theory. This material is standard and details can be found, for example, in [MS12, Chapter 3] and [Wen19a].

Let (M, J) be an almost complex manifold and let $u\colon (\Sigma, j) \to (M, J)$ be a non-constant J–holomorphic map. Set

$$\mathrm{Aut}(\Sigma, j) \coloneqq \{\phi \in \mathrm{Diff}(\Sigma) : \phi_* j = j\} \quad \text{and} \quad \mathfrak{aut}(\Sigma, j) \coloneqq \{v \in \mathrm{Vect}(\Sigma) : \mathscr{L}_v j = 0\}.$$

Let $\mathscr{J}(\Sigma)$ be the space of almost complex structures on Σ and let $\mathscr{T}(\Sigma) = \mathscr{J}(\Sigma)/\mathrm{Diff}_0(\Sigma)$ be the Teichmüller space. Let $\mathscr{S}$ be a slice of the $\mathrm{Diff}_0(\Sigma)$–action through j; that is: $\mathscr{S}$ is a finite-dimensional $\mathrm{Aut}(\Sigma, j)$–invariant submanifold of $\mathscr{J}(\Sigma)$ containing j and such that the map

$$\begin{aligned} \mathrm{Diff}_0(\Sigma) \times_{\mathrm{Aut}(\Sigma,j)} \mathscr{S} &\to \mathscr{T}(\Sigma), \\ [\phi, k] &\mapsto [\phi_* k] \end{aligned}$$

is a homeomorphism. Denote by

$$\mathrm{d}_{u,j}\bar{\partial}_J\colon \Gamma(u^*TM) \oplus T_j\mathscr{S} \to \Omega^{0,1}(u^*TM)$$

the linearization of the map $(u, j) \mapsto \bar{\partial}_J(u, j)$ restricted to $C^\infty(\Sigma, M) \times \mathscr{S}$ at at (u, j). The action of $\mathrm{Aut}(\Sigma, j)$ on $C^\infty(\Sigma, M) \times \mathscr{S}$ preserves $\bar{\partial}_J^{-1}(0)$. Therefore, there is an inclusion $\mathfrak{aut}(\Sigma, j) \hookrightarrow \ker \mathrm{d}_{u,j}\bar{\partial}_J$. The moduli space of J–holomorphic maps up to reparametrization containing $[u, j]$ has virtual dimension

$$\mathrm{index}\, \mathrm{d}_{u,j}\bar{\partial}_J - \dim \mathfrak{aut}(\Sigma, j) = (n-3)\chi(\Sigma) + 2\langle [\Sigma], u^*c_1(M, J)\rangle.$$

Definition 2.1.3. Let (M, J) be an almost complex manifold of dimension $2n$. The **index** of a J–holomorphic map $u\colon (\Sigma, j) \to (M, J)$ is

$$\mathrm{index}(u) \coloneqq (n-3)\chi(\Sigma) + 2\langle [\Sigma], u^*c_1(M, J)\rangle. \tag{2.1.4}$$

•

Infinitesimal deformations of j do not affect $\mathrm{im}\, u$. Therefore, we restrict our attention to $\mathfrak{d}_{u,J}\colon \Gamma(u^*TM) \to \Omega^{0,1}(\Sigma, u^*TM)$, the restriction of $\mathrm{d}_{u,j}\bar{\partial}_J$ to $\Gamma(u^*TM)$. A brief computation shows that

$$\mathfrak{d}_{u,J}\xi = \frac{1}{2}(\nabla\xi + J \circ (\nabla\xi) \circ j + (\nabla_\xi J) \circ \mathrm{d}u \circ j). \tag{2.1.5}$$

Here ∇ denotes any torsion-free connection on TM and also the induced connection on u^*TM. If (u, j) is a J–holomorphic map, then the right-hand side of (2.1.5) does not depend on the choice of ∇; see [MS12, Proposition 3.1.1]. The operator $\mathfrak{d}_{u,J}$ has the property that if $\xi \in \Gamma(T\Sigma)$, then $\mathfrak{d}_{u,J}(\mathrm{d}u(\xi))$ is a $(0, 1)$–form taking values in $\mathrm{d}u(T\Sigma) \subset u^*TM$. If u is non-constant, then there is a unique complex subbundle

$$Tu \subset u^*TM$$

of rank one containing $\mathrm{d}u(T\Sigma)$ [IS99, Section 1.3]; see also [Wen10, Section 3.3] and Section 2.A for a detailed discussion. Since Tu agrees with $\mathrm{d}u(T\Sigma)$ outside finitely many points, $\mathfrak{d}_{u,J}$ maps $\Gamma(Tu)$ to $\Omega^{0,1}(\Sigma, Tu)$. Infinitesimal deformations along $\Gamma(Tu)$ also do not affect $\operatorname{im} u$. This leads us to the following.

Definition 2.1.6. Let (M, J) be an almost complex manifold. Let $u\colon (\Sigma, j) \to (M, J)$ be a non-constant J–holomorphic map. Set

$$Nu := u^*TM/Tu.$$

The **normal Cauchy–Riemann operator** associated with u is the linear map

$$\mathfrak{d}^N_{u,J}\colon \Gamma(Nu) \to \Omega^{0,1}(\Sigma, Nu)$$

induced by $\mathfrak{d}_{u,J}$. •

The following illuminates the role of the normal Cauchy–Riemann operator in the infinitesimal deformation theory of J–holomorphic maps.

Proposition 2.1.7 ([IS99, Lemma 1.5.1; Wen10, Theorem 3]; see also Section 2.A). *Let (M, J) be an almost complex manifold. Let $u\colon (\Sigma, j) \to (M, J)$ be a non-constant J–holomorphic map. Denote by $Z(\mathrm{d}u)$ the number of critical points of u counted with multiplicity. The following hold:*

(1) *There is a surjection*

$$\ker \mathrm{d}_{u,j}\bar\partial_J \twoheadrightarrow \ker \mathfrak{d}^N_{u,J}$$

whose kernel contains $\mathfrak{aut}(\Sigma, j)$ and has dimension $\dim \mathfrak{aut}(\Sigma, j) + 2Z(\mathrm{d}u)$.[11]

(2) *There is an isomorphism*

$$\operatorname{coker} \mathrm{d}_{u,j}\bar\partial_J \cong \operatorname{coker} \mathfrak{d}^N_{u,J}.$$

(3) *The index of $\mathfrak{d}^N_{u,J}$ satisfies*

$$\operatorname{index} \mathfrak{d}^N_{u,J} = \operatorname{index}(u) - 2Z(\mathrm{d}u) \leqslant \operatorname{index}(u).$$

Finally, everything is in place to define super-rigidity.

Definition 2.1.8. Let (M, J) be an almost complex manifold. A non-constant J–holomorphic map u is **rigid** if $\ker \mathfrak{d}^N_{u,J} = 0$. •

A multiple cover $\tilde u$ of u may fail to be rigid, even if u itself is rigid.

Definition 2.1.9. Let (M, J) be an almost complex manifold. A simple J–holomorphic map $u\colon (\Sigma, j) \to (M, J)$ is called **super-rigid** if it is rigid and all of its multiple covers are rigid. •

[11] The summand $2Z(\mathrm{d}u)$ corresponds to infinitesimally deforming the location of the critical points of u without deforming $\operatorname{im} u$.

Remark 2.1.10. A generic simple pseudo-holomorphic map u satisfies $\operatorname{index}(u) \geqslant 0$. If u is also a rigid immersion, then Proposition 2.1.7 implies that $\operatorname{index}(u) = 0$. Conversely, for a generic J every simple J–holomorphic map of index zero is a rigid immersion. In light of this, henceforth, we focus on index zero map. ♣

Definition 2.1.11. Let M be a manifold of dimension at least six. An almost complex structure J on M is called **super-rigid** if the following hold:

(1) Every simple J–holomorphic map has non-negative index.

(2) Every simple J–holomorphic map of index zero is an embedding, and every two simple J–holomorphic maps of index zero either have disjoint images or are related by a reparametrization.

(3) Every simple J–holomorphic map of index zero is super-rigid. •

Remark 2.1.12. If $\dim M = 4$, one should weaken condition (2) and require only that every simple J–holomorphic map of index zero is an immersion with transverse self-intersections, and that two such maps are either transverse to one another or are related by a reparametrization. However, we will only be concerned with dimension at least six. ♣

Let (M, ω) be a symplectic manifold. Bryan and Pandharipande [BP01, Section 1.2] conjectured that a generic almost complex structure compatible with ω is super-rigid. This conjecture has recently been proved by Wendl [Wen19a]. This part of the present article is an exposition of Wendl's proof using the theory developed in Part 1

Theorem 2.1.13 (Wendl [Wen19b]). *Let (M, ω) be a closed, connected symplectic manifold with $\dim M \geqslant 6$. Denote by $\mathscr{J}(M, \omega)$ the Fréchet manifold of smooth almost complex structures compatible with ω. The subspace $\mathscr{J}_\blacklozenge(M, \omega) \subset \mathscr{J}(M, \omega)$ of super-rigid almost complex structures is comeager.*

Remark 2.1.14. By the Baire category theorem, $\mathscr{J}_\blacklozenge(M, \omega)$ is dense in $\mathscr{J}(M, \omega)$. Therefore, every $J \in \mathscr{J}(M, \omega)$ can be arbitrarily slightly perturbed into a super-rigid almost complex structure. The analogue is for paths $(J_t)_{t \in [0,1]}$ fails. This is discussed in detail in Section 2.10. ♣

The proof of Theorem 2.1.13 occupies the bulk of the remainder of Part 2. Throughout the remainder of this part, let (M, ω) be a closed, connected symplectic manifold with $\dim M \geqslant 6$.

2.2 Floer's C^ε spaces and Taubes' trick

A technical but important issue in the proof of Theorem 2.1.13 is that $\mathscr{J}(M, \omega)$ is not a Banach manifold; hence, the theory developed in Part 1 cannot directly be brought to bear on it. The initial impulse might be to work with C^k (instead of smooth) almost complex structures. However, Section 1.6 requires the linear elliptic differential operators under consideration to have smooth coefficients. The solution of this conundrum is to work with the C^ε spaces introduced by Floer [Flo88, Section 5] and employ Taubes' trick, which allows one to pass from the C^ε topology to the C^∞ topology; see [Tau96, Section 5] and [Wen19a, Appendix B] for applications to super-rigidity. Wendl's blog post [Wen21] clarifies how to properly use the C^ε spaces. The purpose of this section is to marshal the salient facts required for the proof of Theorem 2.1.13.

Definition 2.2.1. Denote by

$$\mathfrak{s} := (0,\infty)^{\mathbf{N}_0}$$

the set of sequences in $(0,\infty)$. Define the preorder $\preccurlyeq$ on $\mathfrak{s}$ by

$$(\varepsilon_k) \preccurlyeq (\delta_k) \quad \text{if and only if} \quad \limsup_{k\to\infty} \frac{\varepsilon_k}{\delta_k} < \infty. \qquad \bullet$$

The following observation is nearly trivial but crucial.

Proposition 2.2.2. *Every countable subset $\mathfrak{s}_0 \subset \mathfrak{s}$ has a lower bound; that is: there is a $\delta \in \mathfrak{s}$ with $\delta \preccurlyeq \varepsilon$ for every $\varepsilon \in \mathfrak{s}_0$.*

Proof. Enumerate $\mathfrak{s}_0 = \{\varepsilon^\ell : \ell \in \mathbf{N}\}$ and set $\delta_k := \min\{\varepsilon_k^\ell : \ell \in \{1,\ldots,k\}\}$. ∎

Definition 2.2.3. Suppose that a Riemannian metric on M has been chosen. Let E be a Euclidean vector bundle over M equipped with an orthogonal connection. For $\varepsilon = (\varepsilon_k) \in \mathfrak{s}$ and $s \in \Gamma(E)$ set

$$\|s\|_{C^\varepsilon} := \sum_{k=0}^{\infty} \varepsilon_k \|\nabla^k s\|_{C^0}.$$

The vector space

$$C^\varepsilon\Gamma(E) := \{s \in \Gamma(E) : \|s\|_{C^\infty_\varepsilon} < \infty\}$$

equipped with the norm $\|\cdot\|_{C^\varepsilon}$ is a separable Banach space [Wen19a, Theorems B.2, B.5]. •

Proposition 2.2.4. *For every $\varepsilon \in \mathfrak{s}$ the inclusion $C^\varepsilon\Gamma(E) \to \Gamma(E)$ is continuous. Moreover,*

$$\Gamma(E) = \bigcup_{\varepsilon\in\mathfrak{s}} C^\varepsilon\Gamma(E).$$

Proof. It is obvious that the inclusion $C^\varepsilon\Gamma(E) \to \Gamma(E)$ is continuous. If $s \in \Gamma(E)$, then $s \in C^\varepsilon\Gamma(E)$ with $\varepsilon_k := 2^{-k}\|\nabla^k s\|_{C^0}^{-1}$. (Indeed, in light of Proposition 2.2.2, every countable subset of $\Gamma(E)$ is contained in $C^\varepsilon\Gamma(E)$ for some $\varepsilon \in \mathfrak{s}$.) ∎

The above spaces are used as follows. The tangent space to $\mathscr{J}(M,\omega)$ at J_0 is given by

$$T_{J_0}\mathscr{J}(M,\omega) = \left\{\hat{J} \in \Gamma(\mathrm{End}(TM)) : \hat{J}J_0 + J_0\hat{J} = 0 \text{ and } \omega(\hat{J}\cdot,\cdot) + \omega(\cdot,\hat{J}\cdot) = 0\right\}.$$

This means that $T_{J_0}\mathscr{J}$ consists of anti-linear endomorphisms which are skew-adjoint with respect to ω. There is a $\delta > 0$ (independent of J_0) such that the map

$$\exp_{J_0} \colon \left\{\hat{J} \in T_{J_0}\mathscr{J}(M,\omega) : \|\hat{J}\|_{C^0} < \delta\right\} \to \mathscr{J}(M,\omega)$$

defined by

$$\exp_{J_0}(\hat{J}) := (\mathbf{1} + \tfrac{1}{2}J_0\hat{J})J_0(\mathbf{1} + \tfrac{1}{2}J_0\hat{J})^{-1}.$$

is a diffeomorphism.

Definition 2.2.5. For $\varepsilon \in \mathfrak{s}$ define the Banach manifold $\mathscr{U}(M,\omega;J_0,\varepsilon)$ by

$$\mathscr{U}(M,\omega;J_0,\varepsilon) := \left\{\exp_{J_0}(\hat{J}) : \|\hat{J}\|_{C^\varepsilon} < \delta\right\}$$

covered by a single chart $\exp_{J_0}(\hat{J}) \mapsto \hat{J}$. •

For every $J_0 \in \mathscr{J}$ and $\varepsilon \in \mathfrak{s}$ the inclusion $\mathscr{U}(M,\omega;J_0,\varepsilon) \subset \mathscr{J}(M,\omega)$ is continuous (but by no means open).

An almost complex structure $J \in \mathscr{J}(M,\omega)$ fails to be super-rigid if there is a simple J–holomorphic map violating one of the conditions in Definition 2.1.11. The equivalence classes of theses offending pseudo-holomorphic maps form a topological space $\mathscr{M}^\times$ with a canonical projection map $\Pi\colon \mathscr{M}^\times \to \mathscr{J}(M,\omega)$. Therefore, proving Theorem 2.1.13 amounts to establishing that $\operatorname{im}\Pi$ is meager. $\mathscr{M}^\times$ itself stands little chance to be Banach manifold, but its restriction to $\mathscr{U}(M,\omega;J_0,\varepsilon)$ does. The upcoming definition and proposition extend Definition 1.B.1 and Proposition 1.B.4; the latter is an abstract version of Taubes' trick mentioned earlier.

Definition 2.2.6. Let X be a topological space. Let $c \in \mathbf{N}_0$. A subset $S \subset X$ has **codimension at least** c if there are:

(1) a preordered set $(A,\preccurlyeq)$ such that every countable subset $B \subset A$ has a lower bound; that is: there is an $\alpha \in A$ with $\alpha \preccurlyeq \beta$ for every $\beta \in B$;

(2) for every $x_0 \in X$ and $\alpha \in A$ a subset $U_\alpha(x_0) \subset X$ with $x_0 \in U_\alpha(x_0)$ and the structure of a C^1 Banach manifold on the set $U_\alpha(x_0)$ such that the inclusion $U_\alpha(x_0) \subset X$ is continuous;

(3) a metrizable topological space Z and a continuous map $\zeta\colon Z \to X$ such that

$$S \subset \operatorname{im}\zeta$$

and the following conditions hold:

(a) The map ζ is σ–proper; that is: there is a countable cover $Z = \bigcup_{k\in\mathbf{N}} Z_k$ such that for every $k \in \mathbf{N}$ the restriction $\zeta|_{Z_k}$ is proper.

(b) For every $x_0 \in X$ the fiber $\zeta^{-1}(x_0) \subset Z$ is separable.

(c) For every $z_0 \in \zeta^{-1}(x_0)$ there is a $\beta = \beta(z_0) \in A$ and for every $\alpha \in A$ with $\alpha \preccurlyeq \beta$ there is an open subset

$$V_\alpha(z_0) \subset \zeta^{-1}(U_\alpha(x_0))$$

which contains z_0 and has the structure of a Banach orbifold such that the map $\zeta|_{V_\alpha(z_0)}\colon V_\alpha(z_0) \to U_\alpha(x_0)$ is C^1 and $\operatorname{index} \mathrm{d}_z\zeta \leqslant -c$ for every $z \in V_\alpha(z_0)$. •

Proposition 2.2.7. *Let X be a completely metrizable topological space. If $S \subset X$ has codimension at least one, then S is meager.*

Proof. It suffices to prove that $\zeta(Z_k)$ is closed and has empty interior. To prove that $\zeta(Z_k)$ is closed, observe that a proper map between metrizable topological spaces is closed. (Indeed, it suffices that the codomain is metrizable [Pal70].)

To prove that $\zeta(Z_k)$ has empty interior, let $x_0 \in \zeta(Z_k)$. The task at hand is to exhibit a sequence (x_n) that avoids $\zeta(Z_k)$ but converges to x_0. Choose a countable dense subset $\{z_m : m \in \mathbf{N}\} \subset \zeta^{-1}(x_0)$. Let α be a lower bound of $\{\beta(\zeta_m) : m \in \mathbf{N}\}$. The subset

$$W_\alpha \coloneqq \bigcup_{m\in\mathbf{N}} V_\alpha(z_m) \subset \zeta^{-1}(U_\alpha(x_0))$$

is open and contains $\zeta_\alpha^{-1}(x_0)$. By Proposition 1.B.4, the subset $\zeta(W_\alpha) \subset U_\alpha(x_0)$ is meager. By the Baire category theorem, $\zeta(W_\alpha)$ is nowhere-dense in $U_\alpha(x_0)$ and there is a sequence (x_n) that avoids $\zeta(W_\alpha)$ but converges to x_0 in $U_\alpha(x_0)$. Since the inclusion $U_\alpha(x_0) \subset X$ is continuous, (x_n) converges to x_0 in X.

It remains to prove that $x_n \notin \zeta(Z_k)$ provided $n \gg 1$. If not, then after passing to a subsequence there is a sequence (z_n) in Z_k with $\zeta(z_n) = x_n$. Since $\zeta|_{Z_k}$ is proper, after passing to a further subsequence (z_n) converges to a limit $z_0 \in Z_k$ with $\zeta(z_0) = x_0$. Since W_α is open in $\zeta^{-1}(U_\alpha(x_0))$, for $n \gg 1$, $z_n \in W_\alpha$; hence: $x_n \in \zeta(W_\alpha)$—a contradiction. ∎

2.3 Flexibility

The following observation together with Proposition 2.2.2 implies that the various notions of flexibility introduced in Part 1 are satisfied. The reader might find it helpful at this point to review Definition 1.1.9 and to keep in mind that the normal Cauchy–Riemann operator of a J–holomorphic map u is an operator $\Gamma(E) \to \Gamma(F)$ where $E = Nu$ and $F = \overline{\mathrm{Hom}}_{\mathbf{C}}(T\Sigma, Nu)$ is the bundle of complex anti-linear maps from $T\Sigma$ to Nu.

Lemma 2.3.1. *Let $J_0 \in \mathscr{J}(M,\omega)$. Let $u\colon (\Sigma, j) \to (M, J)$ be a simple J_0–holomorphic map. Consider the set of injective points*

$$U \coloneqq \{x \in \Sigma : u^{-1}(u(x)) = \{x\} \text{ and } \mathrm{d}_x u \neq 0\}.$$

For every

$$A \in \Gamma(\mathrm{Hom}(Nu, \overline{\mathrm{Hom}}_{\mathbf{C}}(T\Sigma, Nu))$$

with support in U there are $\varepsilon \in \mathfrak{s}$, $T > 0$, and a path of compatible almost complex structures $(J_t)_{t\in(-T,T)}$ through J_0 in $\mathscr{U}(M,\omega; J_0, \varepsilon)$ such that:

1. *u is J_t–holomorphic for every $t \in (-T, T)$, and*
2. *$\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \mathfrak{d}^N_{u,J_t}\xi = A\xi$ for every $\xi \in \Gamma(Nu)$.*

Proof. As discussed in Section 2.2,

$$T_{J_0}\mathscr{J}(M,\omega) = \left\{\hat{J} \in \Gamma(\mathrm{End}(TM)) : \hat{J}J_0 + J_0\hat{J} = 0 \text{ and } \omega(\hat{J}\cdot,\cdot) + \omega(\cdot,\hat{J}\cdot) = 0\right\}.$$

For $x \in U$, T_xM decomposes as $T_xM = T_x\Sigma \oplus N_x\Sigma$. Given $a \in \Gamma(\overline{\mathrm{Hom}}_{\mathbf{C}}(T\Sigma, Nu))$, denote by $a^\dagger$ its adjoint with respect to ω and set

$$\hat{J} \coloneqq \begin{pmatrix} 0 & -a^\dagger \\ a & 0 \end{pmatrix}.$$

By construction $\hat{J}J + J\hat{J} = 0$ and $\omega(\hat{J}\cdot,\cdot) + \omega(\cdot,\hat{J}\cdot) = 0$; that is: $\hat{J} \in T_J\mathcal{J}(M,\omega)$.

Given $A \in \Gamma(\mathrm{Hom}(Nu, \overline{\mathrm{Hom}}_{\mathbf{C}}(T\Sigma, Nu))$ with support in U, pick $\hat{J} \in T_{J_0}\mathcal{J}(M,\omega)$ with $\hat{J}|_{u(\Sigma)} = 0$ and such that for every $\xi \in \Gamma(Nu)$

$$\frac{1}{2}\nabla_\xi \hat{J} = \begin{pmatrix} 0 & (A(\xi)j)^\dagger \\ -A(\xi)j & 0 \end{pmatrix}.$$

Let $T \ll 1$ and define $(J_t)_{t\in(-T,T)}$ by

$$J_t := \exp_{J_0}(t\hat{J})$$

By construction u is J_t–holomorphic for every $t \in (-T,T)$. It follows from (2.1.5) that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathfrak{d}^N_{u,J_t}\xi = A\xi.$$

for every $\xi \in \Gamma(Nu)$. Evidently, $(J_t)_{t\in(-T,T)}$ is a path in $\mathscr{U}(M,\omega;J_0,\varepsilon)$ provided $\varepsilon \in \mathfrak{s}$ decays sufficiently fast. ■

2.4 Unobstructedness and embeddedness

The purpose of this section is to take care of (1) and (2) in Definition 2.1.11.

Definition 2.4.1. Let $k \in \mathbf{Z}$. The **universal moduli space of simple J–holomorphic maps of index k over $\mathcal{J}(M,\omega)$** is the space $\mathscr{M}_k(M,\omega)$ of pairs $(J;[u,j])$ consisting of an almost complex structure $J \in \mathcal{J}(M,\omega)$, and an equivalence class of simple J–holomorphic maps $u\colon (\Sigma,j) \to (M,J)$ of index k up to reparametrization by $\mathrm{Diff}(\Sigma)$. •

Theorem 2.4.2. *The subset*

$$\mathscr{W}_{\geqslant 0}(M,\omega) := \{J \in \mathcal{J}(M,\omega) : (1) \text{ in Definition 2.1.11 fails}\}$$

has codimension at least two (in the sense of Definition 2.2.6).

At its core this is a standard transversality result for simple J–holomorphic maps; cf. [MS12, Theorem 3.1.5 (ii)]. Let us spell out its proof nevertheless because it illuminates Definition 2.2.6 and serves as a model for proofs of other parts of the super-rigidity theorem.

Proof of Theorem 2.4.2. The task at hand is to verify the conditions in Definition 2.2.6 for

$$A := \mathfrak{s}, \quad U_\varepsilon(J_0) := \mathscr{U}(M,\omega;J_0,\varepsilon), \quad Z := \coprod_{k<0} \mathscr{M}_k(M,\omega), \quad \text{and}$$

with $\zeta := \Pi$ denoting the projection map. Indeed, $\mathscr{W}_{\geqslant 0}(M,\omega) = \operatorname{im}\Pi$.

Proposition 2.2.2 implies (1). (2) holds by construction. The fact that Π is σ–proper (3.a) is a consequence of the fact that quantitative bounds on the underlying Riemann surface (Σ,j) and the map u guarantee compactness; see [MS12, Proof of Theorem 3.1.5 (ii)] for details. The fact that the fibres $\Pi^{-1}(J_0)$ are separable is standard.

Here is the crucial point: establishing (3.c). A neighborhood of $(J_0, [u_0, j_0]) \in \Pi^{-1}(\mathscr{U}(M,\omega; J_0, \varepsilon))$ is given by

$$\mathscr{F}^{-1}(0)/\mathrm{Aut}(\Sigma_0, j_0).$$

Here $\mathscr{F}$ denotes the restriction of

$$\begin{aligned} W^{1,p}\Gamma(u_0^*TM) \oplus T_{j_0}\mathcal{S} \times T_{J_0}\mathscr{U}(M,\omega; J_0, \varepsilon) &\to L^p\Omega^{0,1}(u_0^*TM), \\ (\xi, j; J) &\mapsto \Phi(\xi)^{-1}\bar\partial_J(\exp_{u_0}(\xi), j) \end{aligned}$$

to a sufficiently small $\mathrm{Aut}(\Sigma_0, j_0)$–invariant neigborhood of 0. Here $p > 2$, and

$$\Phi(\xi)\colon\ L^p(u_0^*TM) \to L^p\Gamma(\exp_{u_0}(\xi)^*TM)$$

is the complex bundle isomorphism induced by parallel transport. Since

$$\mathrm{d}_{u_0,j_0}\bar\partial_{J_0}\colon\ W^{1,p}\Gamma(u^*TM) \oplus T_{j_0}\mathcal{S} \to L^p\Omega^{0,1}(u^*TM)$$

is Fredholm, it has finite dimensional cokernel. As a consequence of Proposition 2.2.2 and Lemma 2.3.1, there is a $\delta \in \mathfrak{s}$ such that every $\varepsilon \preccurlyeq \delta$ the linearization of $\mathscr{F}$ at $(0, j_0; J_0)$ is surjective. Hence, by the Regular Value Theorem, there is an open neighborhood $\mathscr{V}([u_0, j_0]; J_0, \varepsilon)$ of $([u_0, j_0]; J_0) \in \Pi^{-1}(\mathscr{U}(M,\omega; J_0, \varepsilon)$ which carries the structure of a Banach orbifold. For every, $([u, j]; J) \in \mathscr{V}([u_0, j_0]; J_0, \varepsilon)$ we have

$$\mathrm{index}\,\mathrm{d}_{[u,j];J}\Pi = \mathrm{index}(u) \leqslant -2,$$

since, by the index formula (2.1.4), $\mathscr{M}_{-1}(M,\omega) = \varnothing$. Therefore, (3.c) holds. ∎

Theorem 2.4.3 ([OZ09, Theorem 1.1; IP18, Proposition A.4]). *The subset*

$$\mathscr{W}_{\hookrightarrow}(M,\omega) := \{J \in \mathscr{J}(M,\omega) : \text{(2) in Definition 2.1.11 fails}\}$$

has codimension at least $2(n-2)$. ∎

2.5 Petri's condition

The objective of the next five sections is to prove that

$$\mathscr{W}_{\blacklozenge}(M,\omega) := \{J \in \mathscr{J}(M,\omega) : \text{(3) in Definition 2.1.11 fails}\}$$

has codimension at least one in the sense of Definition 2.2.6. This will be achieved using the theory developed in Part 1 applied to certain families of elliptic operators which will be introduced in Section 2.6 and Section 2.9. The result of this section, proved by Wendl, ensures that real Cauchy–Riemann operators satisfy the algebraic condition introduced in Definition 1.6.15 and required in Theorem 1.6.17. This will guarantee that in the application the $\mathfrak{B}$–equivariant Petri condition holds away from a subset of infinite codimension.

Definition 2.5.1. Let (Σ, j) be a Riemann surface and let E be a complex vector bundle over Σ. A **real Cauchy–Riemann operator** on E is a real linear first order elliptic differential operator $D\colon \Gamma(E) \to \Omega^{0,1}(\Sigma, E)$ satisfying

$$\mathfrak{d}(fs) = \bar\partial f \otimes_{\mathbf{C}} s + f\mathfrak{d}s \tag{2.5.2}$$

for every $f \in C^\infty(\Sigma, \mathbf{R})$ and $s \in \Gamma(E)$. In the above formula, $\bar\partial f \in \Omega^{0,1}(\Sigma)$ is defined by considering f as a $\mathbf{C}$–valued function. •

By (2.1.5), the normal Cauchy–Riemann operator associated with a J–holomorphic map is a real Cauchy–Riemann operator.

Theorem 2.5.3 (Wendl [Wen19b, Section 5.3]). *Every symbol of a real Cauchy–Riemann operator satisfies Wendl's condition.*

Before embarking on the proof of this result, let us remind the reader of the following fact. Let V and W be complex vector spaces. Denote by $\overline{W}$ the complex vector space W with scalar multiplication $(\lambda, w) \mapsto \bar\lambda w$. The tensor product $V \otimes W$ admits two commuting complex structures: $I_1 := i \otimes \mathbf{1}$ and $I_2 := \mathbf{1} \otimes i$. $V \otimes W$ decomposes into the subspace on which $I_1 = I_2$ and the subspace on which $I_1 = -I_2$. These can be identified with $V \otimes_{\mathbf{C}} W$ and $V \otimes_{\mathbf{C}} \overline{W}$; hence:

$$V \otimes W = (V \otimes_{\mathbf{C}} W) \oplus (V \otimes_{\mathbf{C}} \overline{W}).$$

The space of real linear maps $\mathrm{Hom}(V, W)$ admits two commuting complex structures given by pre- and post-composition with i. This decomposes $\mathrm{Hom}(V, W)$ into the space of complex linear map $\mathrm{Hom}_{\mathbf{C}}(V, W)$ and the space of complex anti-linear maps $\overline{\mathrm{Hom}}_{\mathbf{C}}(V, W)$; that is:

$$\mathrm{Hom}(V, W) = \mathrm{Hom}_{\mathbf{C}}(V, W) \oplus \overline{\mathrm{Hom}}_{\mathbf{C}}(V, W).$$

Proof of Theorem 2.5.3. The symbol

$$\sigma := \sigma_x(\mathfrak{d})$$

at $x \in \Sigma$ of a real Cauchy–Riemann operator $\mathfrak{d}\colon \Gamma(E) \to \Omega^{0,1}(\Sigma, E) = \Gamma(F)$ depends only on E_x. Denote by $z = s + it$ a local holomorphic coordinate around x and identify $E_x = \mathbf{C}^r$ with $r := \mathrm{rk}_{\mathbf{C}} E$. Identifying

$$\mathbf{R}[s,t] \otimes E_x = \mathbf{R}[s,t] \otimes F_x = \mathbf{R}[s,t] \otimes E_x^\dagger = \mathbf{R}[s,t] \otimes F_x^\dagger = \mathbf{C}[z, \bar z] \otimes_{\mathbf{C}} \mathbf{C}^r$$

the formal differential operators $\hat\sigma$ and $-\hat\sigma^\dagger$ both become

$$\bar\partial \otimes_{\mathbf{C}} \mathrm{id}_{\mathbf{C}^r}\colon \ \mathbf{C}[z, \bar z] \otimes_{\mathbf{C}} \mathbf{C}^r \to \mathbf{C}[z, \bar z] \otimes_{\mathbf{C}} \mathbf{C}^r.$$

Furthermore, identifying

$$\mathbf{C}^r \otimes \mathbf{C}^r = (\mathbf{C}^r \otimes_{\mathbf{C}} \mathbf{C}^r) \oplus (\mathbf{C}^r \otimes_{\mathbf{C}} \bar{\mathbf{C}}^r)$$

via $v \otimes w \mapsto (v \otimes_{\mathbf{C}} w, v \otimes_{\mathbf{C}} \bar w)$ the polynomial Petri map $\hat\varpi$ becomes

$$(\hat\varpi_1, \hat\varpi_2)\colon \ (\mathbf{C}[z, \bar z] \otimes_{\mathbf{C}} \mathbf{C}^r)^{\otimes 2} \to \mathbf{C}[z, \bar z] \otimes_{\mathbf{C}} (\mathbf{C}^r \otimes_{\mathbf{C}} \mathbf{C}^r) \oplus \mathbf{C}[z, \bar z] \otimes_{\mathbf{C}} (\mathbf{C}^r \otimes_{\mathbf{C}} \bar{\mathbf{C}}^r)$$

defined by

$$\hat\varpi_1(p, q) := pq \quad \text{and} \quad \hat\varpi_2(p, q) := p\bar q.$$

From this it is evident that it suffices to consider the case $r = 1$ to prove Theorem 2.5.3.

Proposition 2.5.4. *If $B \in \ker \hat{\varpi}_\sigma$ is homogeneous of degree d, then it is of the form*

$$B = \sum_{j=0}^{d} b_j \big(z^j \otimes z^{d-j} - iz^j \otimes iz^{d-j}\big) + b'_j \big(iz^j \otimes z^{d-j} + z^j \otimes iz^{d-j}\big). \tag{2.5.5}$$

with $b, b' \in \mathbf{R}^d$ satisfying

$$\sum_{j=0}^{d} b_j = 0 \quad \textit{and} \quad \sum_{j=0}^{d} b'_j = 0. \tag{2.5.6}$$

Proof. Every homogeneous $B \in \ker \bar{\partial} \otimes \ker \bar{\partial}$ of degree d is of the form

$$B = \sum_{j=0}^{d} b_j z^j \otimes z^{d-j} + b'_j iz^j \otimes z^{d-j} + b''_j z^j \otimes iz^{d-j} + b'''_j iz^j \otimes iz^{d-j}$$

with $b, b', b'', b''' \in \mathbf{R}^d$. B satisfies $\hat{\varpi}_2(B) = 0$ if and only if for every $j = 0, \dots, d$

$$b_j + b'''_j = 0 \quad \text{and} \quad b'_j - b''_j = 0;$$

that is: B is of the form (2.5.5). If B is of this form, then $\hat{\varpi}_1(B) = 0$ is equivalent to (2.5.6). ∎

Henceforth, let $B \in \ker \varpi_\sigma$ be homogeneous of degree d. The right-inverse of $\hat{\sigma} (= -\hat{\sigma}^\dagger)$ can be chosen as

$$\hat{R}(z^\alpha \bar{z}^\beta) = \frac{1}{\beta + 1} z^\alpha \bar{z}^{\beta+1}. \tag{2.5.7}$$

Define the map $\hat{\mathbf{L}}_{\sigma,B} \colon \mathbf{C}[z,\bar{z}] \otimes_{\mathbf{C}} \mathrm{Hom}(\mathbf{C},\mathbf{C}) \to \mathbf{C}[z,\bar{z}] \otimes_{\mathbf{C}} (\mathbf{C} \otimes \mathbf{C})$ by

$$\hat{\mathbf{L}}_{\sigma,B}(A) \coloneqq \hat{\varpi}\big((\hat{R}A \otimes \mathbf{1} + \mathbf{1} \otimes \hat{R}^\dagger A^\dagger)B\big)$$

as in Definition 1.6.15. $\mathrm{Hom}(\mathbf{C},\mathbf{C})$ and $\mathbf{C} \otimes \mathbf{C}$ decompose as

$$\mathrm{Hom}(\mathbf{C},\mathbf{C}) = \mathrm{Hom}_{\mathbf{C}}(\mathbf{C},\mathbf{C}) \oplus \overline{\mathrm{Hom}}_{\mathbf{C}}(\mathbf{C},\mathbf{C}) \quad \text{and} \quad \mathbf{C} \otimes \mathbf{C} = (\mathbf{C} \otimes_{\mathbf{C}} \mathbf{C}) \oplus (\mathbf{C} \otimes_{\mathbf{C}} \overline{\mathbf{C}}).$$

This induces decompositions of the domain and codomain of $\hat{\mathbf{L}}_{\sigma,B}$. Each of the summands is isomorphic to $\mathbf{C}[z,\bar{z}]$. With respect to these decompositions, $\hat{\mathbf{L}}_{\sigma,B}$ is a matrix of four operators $\mathbf{C}[z,\bar{z}] \to \mathbf{C}[z,\bar{z}]$. Denote by

$$\mathbf{Q}_B \colon \mathbf{C}[z,\bar{z}] \to \mathbf{C}[z,\bar{z}]$$

the bottom right component of $\hat{\mathbf{L}}_{\sigma,B}$ that is: the restriction of $\hat{\mathbf{L}}_{\sigma,B}$ to $\mathbf{C}[z,\bar{z}] \otimes_{\mathbf{C}} \overline{\mathrm{Hom}}_{\mathbf{C}}(\mathbf{C},\mathbf{C})$ composed with the projection to $\mathbf{C}[z,\bar{z}] \otimes_{\mathbf{C}} (\mathbf{C} \otimes_{\mathbf{C}} \overline{\mathbf{C}})$.[12] (The other components of $\hat{\mathbf{L}}_{\sigma,B}$ can be

[12]This operator is different from the one in [Wen19b, Section 5.3]. The origin of this difference is that Wendl defines the formal adjoint $\mathfrak{d}^*$ using the Hermitian metric in contrast to our definition of $\mathfrak{d}^\dagger$ in Definition 1.1.10. Both operators are related by $\mathfrak{d}^\dagger = \bar{*} \circ \mathfrak{d}^* \circ \bar{*}$ with $\bar{*} \colon \Lambda^{p,q} T^*\Sigma \otimes_{\mathbf{C}} E \to \Lambda^{1-p,1-q} T^*\Sigma \otimes_{\mathbf{C}} E^*$ denoting the anti-linear Hodge star operator.

seen to vanish.) For $\ell \in \mathbf{N}_0$ denote by $\mathbf{C}[z,\bar{z}]^{\leqslant \ell}$ the ring of polynomials in z and $\bar{z}$ of degree at most ℓ and denote by $\mathbf{Q}^{\leqslant \ell}\colon \mathbf{C}[z,\bar{z}]^{\leqslant \ell} \to \mathbf{C}[z,\bar{z}]^{\leqslant \ell+1}$ the truncation of $\mathbf{Q}_B$. The map $\mathbf{Q}_B$ is complex linear. Since

$$\operatorname{rk} \hat{\mathbf{L}}_{\sigma,B}^{\leqslant \ell} \geqslant \operatorname{rk}_{\mathbf{C}} \mathbf{Q}_B^{\leqslant \ell},$$

it suffices to estimate the latter.

Proposition 2.5.8. *The map* $\mathbf{Q}_B$ *satisfies*

$$\operatorname{rk}_{\mathbf{C}} \mathbf{Q}_B^{\leqslant \ell} \geqslant \frac{1}{16}\ell^2$$

for $\ell \geqslant 8d$.

Proof. The map $\mathbf{Q}_B$ can be computed explicitly. To do so, observe that $\overline{\mathrm{Hom}_{\mathbf{C}}}(\mathbf{C},\mathbf{C}) = \mathbf{C}$ acts on $\mathbf{C}$ via $\lambda \cdot \mu := \lambda\bar{\mu}$ and its adjoint is $\lambda^\dagger \cdot \mu := \bar{\lambda}\bar{\mu}$; furthermore, recall that $\hat{R}^\dagger = -\hat{R}$ and that $\hat{R}$ is given by (2.5.7). With this in mind it is easy to verify that for

$$A = z^\alpha \bar{z}^\beta \quad \text{and} \quad B = \sum_{j=0}^{d} b_j\big(z^j \otimes z^{d-j} - iz^j \otimes iz^{d-j}\big) + b_j'\big(iz^j \otimes z^{d-j} + z^j \otimes iz^{d-j}\big)$$

the map $\mathbf{Q}_B$ satisfies

$$\begin{aligned}
\mathbf{Q}_B(A) &= \hat{\omega}_2\big((\hat{R}A \otimes \mathbf{1} + \mathbf{1} \otimes \hat{R}^\dagger A^\dagger)B\big)\\
&= \sum_{j=0}^{d} 2(b_j - ib_j')\hat{R}(z^\alpha \bar{z}^{\beta+j})\bar{z}^{d-j} - 2(b_j + ib_j')z^j\overline{\hat{R}(z^\beta \bar{z}^{\alpha+d-j})}\\
&= \sum_{j=0}^{d} \frac{2(b_j - ib_j')}{\beta + j + 1} z^\alpha \bar{z}^{\beta+d+1} - \frac{2(b_j + ib_j')}{\alpha + d - j + 1} z^{\alpha+d+1}\bar{z}^\beta\\
&= \Big(p_\beta^B \bar{z}^{d+1} + q_\alpha^B z^{d+1}\Big) z^\alpha \bar{z}^\beta
\end{aligned}$$

with

$$p_\beta^B := \sum_{j=0}^{d} \frac{2(b_j - ib_j')}{\beta + j + 1}\bar{z}^{d+1} \quad \text{and} \quad q_\alpha^B := -\sum_{j=0}^{d} \frac{2(b_j + ib_j')}{\alpha + d - j + 1} z^{d+1}.$$

The same formula holds for $\mathbf{Q}_B^{\leqslant \ell}$ provided $\alpha + \beta + d \leqslant \ell$.

Set

$$S := \Big\{(\alpha,\beta) \in \mathbf{N}_0^2 : \alpha + \beta + d \leqslant \ell \text{ and } (p_\beta^B, q_\alpha^B) \neq (0,0)\Big\}.$$

Choose a subset $S^\star \subset S$ such that $\#S^\star \geqslant \frac{1}{2}\#S$ and such that if $(\alpha,\beta) \in S$, then $(\alpha - d - 1, \beta + d + 1) \notin S$. The restriction of $\mathbf{Q}_B^{\leqslant \ell}$ to $\mathbf{C}\langle z^\alpha \bar{z}^\beta : (\alpha,\beta) \in S^\star\rangle$ is injective. The latter is evident from the construction of $S^\star$ and

$$\mathbf{Q}_B\left(\sum_{\alpha,\beta \in S^\star} \lambda_{\alpha,\beta} z^\alpha \bar{z}^\beta\right) = \sum_{\alpha,\beta \in S^\star} \lambda_{\alpha,\beta}\Big(p_\beta^B z^\alpha \bar{z}^{\beta+d+1} + q_\alpha^B z^{\alpha+d+1}\bar{z}^\beta\Big).$$

Therefore,

$$\mathrm{rk}_{\mathbf{C}}\,\mathbf{Q}_B \geqslant \frac{1}{2}\#S.$$

It remains to find a lower bound on $\#S$. At most d of the numbers $p_0^B, p_1^B, \ldots$ are non-zero. This is a consequence of the following. If $\beta_0, \ldots, \beta_{d+1}$ are $d+1$ distinct positive numbers, then the matrix

$$\begin{pmatrix} \frac{1}{\beta_0+1} & \frac{1}{\beta_0+2} & \cdots & \frac{1}{\beta_0+d+1} \\ \frac{1}{\beta_0+1} & \frac{1}{\beta_1+2} & \cdots & \frac{1}{\beta_1+d+1} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{\beta_d+1} & \frac{1}{\beta_d+2} & \cdots & \frac{1}{\beta_d+d+1} \end{pmatrix}$$

is a Cauchy matrix and, therefore, invertible. If $p_{\beta_0}^B = \cdots = p_{\beta_d}^B = 0$, then the product of the above matrix with $(b_0 - ib_0, \ldots, b_d - ib_d)$ would vanish: a contradiction. A variation of this argument shows that at most d of the numbers $q_0^B, q_1^B, \ldots$ are non-zero. Therefore,

$$\#S \geqslant \frac{(\ell-d)^2}{4} - (\ell-d)d.$$

This implies the assertion. ∎

This finishes the proof of Theorem 2.5.3 with $c_0(\rho, d) = \frac{1}{16}$ and $\ell_0(\rho, d) = 8d$. ∎

2.6 Rigidity of unbranched covers

The purpose of this section is to prove the following.

Proposition 2.6.1 (cf. [GW17, Theorem 1.3]). *Denote by $\mathscr{W}_\Diamond(M,\omega)$ the subset of those $J \in \mathscr{J}(M,\omega)$ for which there is a simple J–holomorphic map u of index zero such that an* unbranched *cover of u fails to be rigid. $\mathscr{W}_\Diamond(M,\omega)$ has codimension at least one (in the sense of Definition 2.2.6).*

This is a only warm-up because it does not account for branched covers. However, it is instructive to see the proof in the special case as it will be a model for the general case. The following discussion puts us in a position to prove Proposition 2.6.1 using Theorem 1.3.5.

Definition 2.6.2. Let (Σ, j) and $(\tilde\Sigma, \tilde j)$ be Riemann surfaces, let E be a complex vector bundle over Σ, and let $\mathfrak{d}\colon \Gamma(E) \to \Omega^{0,1}(\Sigma, E)$ be a real Cauchy–Riemann operator. Let $\pi\colon (\tilde\Sigma, j) \to (\Sigma, j)$ be a non-constant holomorphic map. The **pullback** of $\mathfrak{d}$ by π is the real Cauchy–Riemann operator

$$\pi^\star\mathfrak{d}\colon \Gamma(\pi^*E) \to \Omega^{0,1}(\tilde\Sigma, \pi^*E)$$

characterized by

$$(\pi^\star\mathfrak{d})(\pi^*s) = \pi^*(\mathfrak{d}s).$$ •

The following is proved as Proposition 2.A.3 in Section 2.A.

Proposition 2.6.3. *Let $u\colon (\Sigma, j) \to (M, J)$ be a non-constant J–holomorphic map. If $\pi\colon (\tilde\Sigma, \tilde j) \to (\Sigma, j)$ is a non-constant holomorphic map and $\tilde u \coloneqq u \circ \pi$, then there is an isomorphism $N(u\circ\pi) \cong \pi^*Nu$ with respect to which $\mathfrak{d}^N_{u\circ\pi,J} = \pi^\star\mathfrak{d}^N_{u,J}$.*

In the situation of Definition 2.6.2, if π is covering map, then $\pi^\star\mathfrak{d}$ and $\pi^*\mathfrak{d}$ defined in Definition 1.2.1 are related by the commutative diagram

$$\begin{tikzcd}
\Gamma(\pi^*E) \arrow[r, "\pi^*\mathfrak{d}"] \arrow[d, equal] & \Gamma(\pi^*T^*\Sigma^{0,1} \otimes_{\mathbf{C}} \pi^*E) \arrow[d, "\pi^*"] \\
\Gamma(\pi^*E) \arrow[r, "\pi^\star\mathfrak{d}"] & \Omega^{0,1}(\tilde{\Sigma}, \pi^*E)
\end{tikzcd}$$

with π^* being an isomorphism. (This explains the intentionally confusing choice of notation.) Therefore and by Proposition 1.2.9, if $J \in \mathscr{W}_\Diamond$, then there exists a simple J–holomorphic map $u\colon (\Sigma, j) \to (M, J)$ of index zero and an irreducible Euclidean local system $\underline{V}$ whose monodromy representation factors through a finite quotient of $\pi_1(\Sigma, x_0)$ such that

$$\ker \mathfrak{d}_{u,J}^{\underline{V}} \neq 0.$$

Proof of Proposition 2.6.1. As in the proof of Theorem 2.4.2, the only nontrivial part is to verify condition (3.c) from Definition 2.2.6. To that end, let $(J_0, [u_0, j_0]) \in \mathscr{M}_0(M, \omega)$ and let $\varepsilon \in \mathfrak{s}$ be a sequence converging to zero. By Theorem 2.4.3 it suffices to consider the case when $u_0 \colon \Sigma \to M$ is an embedding. Let $\mathscr{V}$ be an open neighborhood of $(J_0, [u_0, j_0])$ in the universal moduli space of simple maps over $\mathscr{U} = \mathscr{U}(M, \omega; J_0, \varepsilon) \subset \mathscr{J}(M, \omega)$ with the following two properties. First, for every $(J, [u, j]) \in \mathscr{V}$ the map u is an embedding. Second, $\mathscr{V}$ is **liftable** in the sense that there is an $\mathrm{Aut}(\Sigma, j_0)$–invariant slice $\mathscr{S}$ of Teichmüller space through j_0 such that for every $(J, [u, j]) \in \mathscr{V}$ there is a unique lift $u\colon (\Sigma, j) \to (M, J)$ with $j \in \mathscr{S}$.

Consider the family of normal Cauchy–Riemann operators $\mathfrak{d}_{u,J}^N\colon \Gamma(Nu) \to \Omega^{0,1}(Nu)$ parametrized by $\mathscr{V}$. Strictly speaking, this is not a family of linear elliptic differential operators as in Definition 1.1.1. Indeed, Nu and $\overline{\mathrm{Hom}}_{\mathbf{C}}(T\Sigma, Nu)$ depend on u and j and thus define vector bundles $\mathbf{E}$ and $\mathbf{F}$ over $\mathscr{V} \times \Sigma$. However, after shrinking $\mathscr{V}$, one can construct isomorphisms $\mathbf{E} \cong \mathrm{pr}_\Sigma^* E$ and $\mathbf{F} \cong \mathrm{pr}_\Sigma^* F$ with $E \coloneqq Nu_0$ and $F \coloneqq \mathrm{Hom}_{\mathbf{C}}(T\Sigma, Nu_0)$ for $(J_0, [u_0, j_0]) \in \mathscr{V}$. Employing these isomorphisms, the normal Cauchy–Riemann operators form a family of linear elliptic differential operators

$$\mathfrak{d}\colon \mathscr{V} \to \mathscr{F}(W^{1,2}\Gamma(E), L^2\Gamma(F))$$

as in Definition 1.1.1. A moment's thought shows that the map $\Lambda_p^{\mathfrak{B}}$ defined in Theorem 1.3.5, $\mathfrak{B}$–equivariant flexiblity defined in Definition 1.3.8, and the $\mathfrak{B}$–equivariant Petri condition defined in Definition 1.3.9 are independent of the choice of isomorphisms $\mathbf{E} \cong \mathrm{pr}_\Sigma^* E$ and $\mathbf{F} \cong \mathrm{pr}_\Sigma^* F$. Therefore, the results from Section 1.3 apply without reservation.

By Lemma 2.3.1, there is $\delta \in \mathfrak{s}$ such that $\mathfrak{d}$ is $\mathfrak{B}$–equivariantly flexible in U provided $\varepsilon \preccurlyeq \delta$. Furthermore, by Theorem 1.6.17 and Theorem 2.5.3, the subset of $(J, [u, j]) \in \mathscr{V}$ for which $\mathfrak{d}_{u,J}^N$ fails to satisfy the $\mathfrak{B}$–equivariant Petri condtion in U has infinite codimension.

Let $\underline{V}$ be an irreducible Euclidean local system on Σ whose monodromy representation factors through a finite quotient of $\pi_1(\Sigma, x_0)$ and consider $\mathfrak{B}$ as in Situation 1.3.1 consisting only of $\underline{V}$. Denote by

$$\mathscr{W}_{\Lambda;\underline{V}}$$

the subset of those $p := (J, [u, j]) \in \mathscr{V}$ for which the map $\Lambda_p^{\mathfrak{B}}$ defined in Theorem 1.3.5 fails to be surjective. Evidently, $\mathscr{W}_{\Lambda;\underline{V}}$ is closed; in particular, $\mathscr{V} \backslash \mathscr{W}_{\Lambda;\underline{V}}$ is a Banach manifold. By the preceding paragraph, $\mathscr{W}_{\Lambda;\underline{V}}$ has infinite codimension. Set

$$\mathscr{W}_{\Diamond;\underline{V}} := \left\{(J, [u, j]) \in \mathscr{V} \backslash \mathscr{W}_{\Lambda;\underline{V}} : \ker \mathfrak{d}_{u,J}^{N,\underline{V}} \neq 0\right\}.$$

Since

$$\operatorname{index} \mathfrak{d}_{u,J}^{N,\underline{V}} = \operatorname{rk} \underline{V} \cdot \operatorname{index} \mathfrak{d}_{u,J}^{N} \leqslant \operatorname{rk} \underline{V} \cdot \operatorname{index}(u) = 0,$$

by Theorem 1.3.5, $\mathscr{W}_{\Diamond;\underline{V}}$ has codimension at least one. Since $(J_0, [u_0, j_0])$ was arbitrary and there are countably many isomorphism classes of irreducible Euclidean local systems, condition (3.c) from Definition 2.2.6 is satisfied.

■

The next two sections develop tools with which the above argument can be carried over to branched covering maps.

2.7 Branched covering maps as orbifold covering maps

An orbifold Riemann surface can be constructed from a smooth Riemann surface and a collection of points equipped with multiplicities.[13] The starting point of this construction is the following observation. Denote by $\mathbf{D}$ the unit disk in $\mathbf{C}$. For $k \in \mathbf{N}$ denote by

$$\mu_k := \{\zeta \in \mathbf{C} : \zeta^k = 1\}$$

the group of k^{th} roots of unity. The map $\pi\colon \mathbf{D} \to \mathbf{D}$ defined by $\pi(z) := z^k$ induces a homeomorphism $\mathbf{D}/\mu_k \cong \mathbf{D}$. Denote by $[\mathbf{D}/\mu_k]$ the orbifold with $\mathbf{D}$ as the underlying topological space and π as chart. The map π also induces an orbifold map $\beta\colon [\mathbf{D}/\mu_k] \to \mathbf{D}$ which induces the identity map on the underlying topological spaces. The identity map $\mathbf{D} \to \mathbf{D}$ defines an orbifold map $\hat{\pi}\colon \mathbf{D} \to [\mathbf{D}/\mu_k]$. This map is a covering map because $\mathbf{D} \cong [(\mathbf{D} \times \mu_k)/\mu_k]$; cf. Footnote 7 on page 11. By construction, $\pi = \beta \circ \hat{\pi}$. This can be globalized as follows.

Definition 2.7.1. Let (Σ, j) be a Riemann surface. A **multiplicity function** is a function $\nu\colon \Sigma \to \mathbf{N}$ such that the set

$$Z_\nu := \{x \in \Sigma : \nu(x) > 1\}$$

is discrete. Given a multiplicity function ν, denote by (Σ_ν, j_ν) the orbifold Riemann surface whose underlying topological space is Σ and such that for every $x \in \Sigma$ and every holomorphic chart $\phi\colon \mathbf{D} \to \Sigma$ with $\phi(0) = x$ the map $\phi_{\nu(x)}\colon \mathbf{D} \to \Sigma$ defined by

$$\phi_{\nu(x)}(z) := \phi(z^{\nu(x)})$$

is a holomorphic orbifold chart. Denote by $\beta_\nu\colon (\Sigma_\nu, j_\nu) \to (\Sigma, j)$ the holomorphic orbifold map given by $z \mapsto z^{\nu(x)}$ with respect to these charts. The underlying continuous map of topological spaces is the identity map $\Sigma \to \Sigma$. •

[13] For an introduction to complex orbifolds we refer the reader to [Kaw79; FS92, Section 1; KM95, Section 8(ii)].

Remark 2.7.2. An orbifold is **effective** if the local stabilizer group of every point acts effectively. Every effective orbifold Riemann surface is isomorphic to one constructed as in Definition 2.7.1. ♣

This construction allows us to canonically associate an orbifold cover with every branched cover of Riemann surfaces.

Proposition 2.7.3. *Let (Σ, j) and $(\tilde{\Sigma}, \tilde{j})$ be Riemann surfaces and let $\pi\colon (\tilde{\Sigma}, \tilde{j}) \to (\Sigma, j)$ be a non-constant holomorphic map. For every $\tilde{x} \in \tilde{\Sigma}$ denote by $r(\tilde{x}) \in \mathbf{N}$ the ramification index of π at $\tilde{x}$. Define $\nu\colon \Sigma \to \mathbf{N}$ and $\tilde{\nu}\colon \tilde{\Sigma} \to \mathbf{N}$ by*

$$\nu(x) \coloneqq \operatorname{lcm}\{r(\tilde{x}) : \tilde{x} \in \pi^{-1}(x)\} \quad \textit{and} \quad \tilde{\nu}(\tilde{x}) \coloneqq \nu(\pi(\tilde{x}))/r(\tilde{x}).$$

Let $(\tilde{\Sigma}_{\tilde{\nu}}, \tilde{j})$ and (Σ_ν, j_ν) be the corresponding orbifold Riemann surfaces constructed in Definition 2.7.1. There is a unique holomorphic covering map $\hat{\pi}\colon (\tilde{\Sigma}_{\tilde{\nu}}, \tilde{j}_{\tilde{\nu}}) \to (\Sigma_\nu, j_\nu)$ such that the diagram

$$\begin{tikzcd}
(\tilde{\Sigma}_{\tilde{\nu}}, \tilde{j}_{\tilde{\nu}}) \arrow[r, "\hat{\pi}"] \arrow[d, "\beta_{\tilde{\nu}}"] & (\Sigma_\nu, j_\nu) \arrow[d, "\beta_\nu"] \\
(\tilde{\Sigma}, \tilde{j}) \arrow[r, "\pi"] & (\Sigma, j)
\end{tikzcd} \tag{2.7.4}$$

commutes.

Proof. For every $x \in \Sigma$ there is a holomorphic chart $\phi\colon \mathbf{D} \to \Sigma$ with $\phi(0) = x$ and for every $\tilde{x} \in \pi^{-1}(x)$ there is a holomorphic chart $\tilde{\phi}\colon \mathbf{D} \to \tilde{\Sigma}$ such that $\tilde{\phi}(0) = \tilde{x}$ and $\pi \circ \phi(z) = \phi(z^{r(\tilde{x})})$. There is a unique orbifold map $\hat{\pi}\colon \tilde{\Sigma}_{\tilde{\nu}} \to \Sigma_\nu$ which is given by the identity map with respect to the charts $\phi_{\tilde{\nu}(\tilde{x})}$ and $\phi_{\nu(x)}$. Evidently, this map is holomorphic. It is a covering map because

$$[\mathbf{D}/\mu_{\tilde{\nu}(\tilde{x})}] \cong [(\mathbf{D} \times_{\mu_{\tilde{\nu}(\tilde{x})}} \mu_{\nu(x)})/\mu_{\nu(x)}]$$

and the canonical map

$$\mathbf{D} \times_{\mu_{\tilde{\nu}(\tilde{x})}} \mu_{\nu(x)} \to \mathbf{D}$$

is a $\mu_{\nu(x)}$–equivariant covering map. ■

Remark 2.7.5. Every covering map of effective orbifold Riemann surfaces arises from a branched cover of the underlying smooth Riemann surfaces by the above construction. ♣

2.8 A criterion for the failure of super-rigidity

Proposition 2.8.3 below shows that the orbifoldization process from Definition 2.7.1 does not affect the kernel and cokernel of real Cauchy–Riemann operators.

Remark 2.8.1. The notion of a real Cauchy–Riemann operators from Definition 2.5.1 can be easily adapted to the orbifold setting. A complex bundle E over an orbifold Riemann surface Σ is given, in a local orbifold chart $\mathbf{D}/\Gamma$, where Γ is the local stabilizer group, by a Γ–equivariant bundle over the disc $\mathbf{D}$. In this chart, sections of E correspond to Γ–equivariant sections over $\mathbf{D}$. A real Cauchy–Riemann operator is a linear map $\mathfrak{d}\colon \Gamma(E) \to \Omega^{0,1}(\Sigma, E)$ which for sections supported in a local orbifold chart agrees with a Γ–equivariant real Cauchy–Riemann operator on $\mathbf{D}$. The pullback construction from Definition 2.6.2 generalizes in an obvious way, by pulling back Γ–equivariant operators in in every orbifold chart. For more details on vector bundles and elliptic operators over orbifolds, see, for example [Kaw79; Kaw81], [SY19, Sections 2-3]; see also [Moe02, Section 5] for a groupoid perspective.[14] ♣

Definition 2.8.2. Let (Σ, j) be a Riemann surface with a multiplicity function ν. Given a complex vector bundle E over Σ and a real Cauchy–Riemann operator $\mathfrak{d}\colon \Gamma(E) \to \Omega^{0,1}(\Sigma, E)$, set

$$E_\nu := \beta_\nu^* E \quad \text{and} \quad \mathfrak{d}_\nu := \beta_\nu^\star \mathfrak{d}$$

with $\beta_\nu^\star \mathfrak{d}\colon \Gamma(E_\nu) \to \Omega^{0,1}(\Sigma_\nu, E_\nu)$ as in Definition 2.6.2. •

Proposition 2.8.3. *If (Σ, j) is a closed Riemann surface with a multiplicity function ν, E is a complex vector bundle over Σ, and $\mathfrak{d}\colon \Gamma(E) \to \Omega^{0,1}(\Sigma, E)$ is a real Cauchy–Riemann operator, then*

$$\ker \mathfrak{d}_\nu \cong \ker \mathfrak{d} \quad \textit{and} \quad \operatorname{coker} \mathfrak{d}_\nu \cong \operatorname{coker} \mathfrak{d}.$$

Proof. The pullback map $\beta_\nu^*\colon \Gamma(E) \to \Gamma(E_\nu)$ induces an injection $\ker \mathfrak{d} \hookrightarrow \ker \mathfrak{d}_\nu$. In fact, this map is an isomorphism. To see this, the following local consideration suffices. Let $x \in \Sigma$ and set $k := \nu(x)$. Define $\beta\colon \mathbf{D} \to \mathbf{D}$ by $\beta(z) := z^k$. Choose a holomorphic chart $\phi\colon \mathbf{D} \to \Sigma$ with $\phi(0) = x$ and a trivialization of E over $\phi(\mathbf{D})$. With respect to these $\mathfrak{d}$ and $\mathfrak{d}_\nu$ can be written as

$$\mathfrak{d} = \bar\partial + \mathfrak{n} \quad \text{and} \quad \mathfrak{d}_\nu = \bar\partial + \beta^*\mathfrak{n}$$

for some $\mathfrak{n} \in \Omega^{0,1}(\mathbf{D}, \operatorname{End}_{\mathbf{R}}(\mathbf{C}^r))$. If $\tilde s \in C^\infty(\mathbf{D}, \mathbf{C}^r)$ is μ_k–invariant, then there is a bounded map $s \in C^\infty(\mathbf{D}\backslash\{0\}, \mathbf{C}^r)$ such that $\tilde s = s \circ \beta$. If $(\bar\partial + \beta^*\mathfrak{n})\tilde s = 0$, then $(\bar\partial + \mathfrak{n})s = 0$; hence, s extends to $\mathbf{D}$ by elliptic regularity.

Since $\operatorname{coker} \mathfrak{d} \cong (\ker \mathfrak{d}^\dagger)^*$ and similarly for $\mathfrak{d}_\nu$, it suffices to produce an isomorphism $\ker \mathfrak{d}^\dagger \cong \ker \mathfrak{d}_\nu^\dagger$. The formal adjoints $\mathfrak{d}^\dagger\colon\ \Omega^{1,0}(\Sigma, E^*) \to \Omega^{1,1}(\Sigma, E^*)$ and $\mathfrak{d}_\nu^\dagger\colon\ \Omega^{1,0}(\Sigma_\nu, E_\nu^*) \to \Omega^{1,1}(\Sigma_\nu, E_\nu^*)$ are real Cauchy–Riemann operators acting on $(1,0)$–forms and locally of the form $\mathfrak{d}^\dagger = \bar\partial + \mathfrak{n}$ and $\mathfrak{d}_\nu^\dagger = \bar\partial + \beta^*\mathfrak{n}$. The pullback map $\beta_\nu^*\colon\ \Omega^{1,0}(\Sigma, E) \to \Omega^{1,0}(\Sigma_\nu, E_\nu)$ induces an injection $\ker \mathfrak{d}^\dagger \hookrightarrow \ker \mathfrak{d}_\nu^\dagger$. This map is an isomorphism by the following local consideration. If $\tilde s \in C^\infty(\mathbf{D}, \mathbf{C}^r)$ is such that $\tilde s\,\mathrm{d}z$ is μ_k–invariant, then there is a map $s \in C^\infty(\mathbf{D}\backslash\{0\}, \mathbf{C}^r)$ such that $\tilde s = kz^{k-1} s \circ \beta$. If $(\bar\partial + \beta^*\mathfrak{n})\tilde s = 0$, then $(\bar\partial + \mathfrak{n})s = 0$ and a consideration of the Taylor expansion of $\tilde s$ shows that s is bounded. Therefore, s extends to $\mathbf{D}$ and $\tilde s\,\mathrm{d}z = \beta^*(s\,\mathrm{d}z)$. ■

This together with the discussion in Section 2.7 leads to the following criterion for the failure of super-rigidity.

Definition 2.8.4. Let (Σ, j) be a Riemann surface with a multiplicity function ν and let $x_0 \in \Sigma\backslash Z_\nu$. For every $x \in Z_\nu$ there is a conjugacy class of a subgroup $\mu_{\nu(x)} \leqslant \pi_1(\Sigma_\nu, x_0)$, generated by the homotopy class of a loop in $\Sigma\backslash Z_\nu$ based at x_0 which is contractible in $(\Sigma\backslash Z_\nu) \cup \{x\}$ and goes around x once. If $\underline{V}$ is a Euclidean local system on Σ_ν, then its **monodromy around** x is the representation $\mu_{\nu(x)} \to \mathrm{O}(V)$ induced by the monodromy representation. •

[14] It is worth keeping in mind that a vector bundle over an orbifold does not have to induce a vector bundle over the underlying topological space; for that reason, some authors prefer the term **orbi-bundle**.

Proposition 2.8.5. *Let $u\colon (\Sigma, j) \to (M, J)$ be a simple J–holomorphic map. If u is not super-rigid, then there are a multiplicity function $\nu\colon \Sigma \to \mathbf{N}$ and an irreducible Euclidean local system $\underline{V}$ on Σ_ν such that:*

(1) *the monodromy representation of $\underline{V}$ factors through a finite quotient of $\pi_1(\Sigma_\nu, x_0)$,*

(2) *$\underline{V}$ has non-trivial monodromy around every point of Z_ν, and*

(3) *the twist*
$$\mathfrak{d}_\nu^{\underline{V}}\colon \Gamma((Nu)_\nu \otimes \underline{V}) \to \Omega^{0,1}(\Sigma_\nu, (Nu)_\nu \otimes \underline{V}) \quad \text{with} \quad \mathfrak{d} := \mathfrak{d}_{u,J}^N$$
has non-trivial kernel.

Proof. Let $(\tilde{\Sigma}, \tilde{j})$ be a closed Riemann surface and $\pi\colon (\tilde{\Sigma}, \tilde{j}) \to (\Sigma, j)$ a non-constant holomorphic map such that $\tilde{u} := u \circ \pi$ is not rigid, that is: $\ker \mathfrak{d}_{\tilde{u},J}^N$ is non-trivial.

Let $\hat{\pi}\colon (\tilde{\Sigma}_{\tilde{\nu}}, \tilde{j}_{\tilde{\nu}}) \to (\Sigma_\nu, j_\nu)$ be the corresponding holomorphic covering map between orbifold Riemann surfaces constructed in Proposition 2.7.3. Set $\mathfrak{d} := \mathfrak{d}_{u,J}^N$ and $\tilde{\mathfrak{d}} := \mathfrak{d}_{u\circ\pi,J}^N$. By Proposition 2.8.3, Proposition 2.6.3, and Proposition 1.2.9,
$$\ker \tilde{\mathfrak{d}} \cong \ker \tilde{\mathfrak{d}}_{\tilde{\nu}} \cong \ker \hat{\pi}^* \mathfrak{d}_\nu \cong \ker \mathfrak{d}_\nu^{\hat{\pi}_* \underline{\mathbf{R}}}.$$
Therefore, $\ker \mathfrak{d}_\nu^{\hat{\pi}_* \underline{\mathbf{R}}}$ is non-trivial.

Since $\hat{\pi}_* \underline{\mathbf{R}}$ decomposes into irreducible local systems, there is an irreducible local system $\underline{V}$ such that $\ker \mathfrak{d}_\nu^{\underline{V}}$ is non-trivial. Define the multiplicity function $\nu'\colon \Sigma \to \mathbf{N}$ by
$$\nu'(x) := \begin{cases} \nu(x) & \text{if } \underline{V} \text{ has non-trivial monodromy around } x \\ 1 & \text{otherwise.} \end{cases}$$
$\underline{V}$ descends to an irreducible local system $\underline{V}'$ on $\Sigma_{\nu'}$ with non-trivial monodromy around every $x \in Z_{\nu'}$. By Proposition 2.8.3, $\ker \mathfrak{d}_{\nu'}^{\underline{V}'} \cong \ker \mathfrak{d}_\nu^{\underline{V}}$. ■

The following index formula is the final preparation required for the proof of Theorem 2.1.13. Its proof is presented in Section 2.B.

Proposition 2.8.6. *Let (Σ, j) be a closed Riemann surface with a multiplicity function ν, let E be a complex vector bundle over Σ, and let $\mathfrak{d}\colon \Gamma(E) \to \Omega^{0,1}(\Sigma, E)$ a real Cauchy–Riemann operator on E. If $\underline{V}$ is a Euclidean local system on Σ_ν, then*
$$\operatorname{index} \mathfrak{d}_\nu^{\underline{V}} = \dim V \operatorname{index} \mathfrak{d} - \operatorname{rk}_{\mathbf{C}} E \sum_{x \in Z_\nu} \dim(V/V^{\rho_x}).$$
Here ρ_x denotes the monodromy of $\underline{V}$ around x, and $V^{\rho_x} \subset V$ is the subspace of ρ_x–invariant vectors.

2.9 The loci of failure of super-rigidity

Denote by $\mathscr{W}_{\blacklozenge}(M, \omega)$ the set of those $J \in \mathscr{J}(M, \omega)$ for which there exists an index zero J–holomorphic embedding $u: (\Sigma, j) \to (M, J)$ which is not super-rigid. To prove Theorem 2.1.13 it remains to prove that $\mathscr{W}_{\blacklozenge}(M, \omega)$ has codimension at least one in the sense of Definition 2.2.6. The proof makes use of the moduli spaces introduced in the following two definitions.

Definition 2.9.1. Let $k \in \mathbf{Z}$ and $s \in \mathbf{N}_0$. Denote by $\mathscr{M}\mathscr{O}_{k,s}(M, \omega)$ the space of pairs $(J, [u, j; \nu])$ consisting of an almost complex structure $J \in \mathscr{J}(M, \omega)$, and an equivalence class $[u, j; \nu]$ of

(1) a simple J–holomorphic map $u: (\Sigma, j) \to (M, J)$ of index k, and

(2) a multiplicity function $\nu: \Sigma \to \mathbf{N}$ with $\# Z_\nu = s$

up to reparametrization by $\mathrm{Diff}(\Sigma)$. The set of multiplicity functions ν is an infinite cover of the regular subset set of the symmetric product $\mathrm{Sym}^s \Sigma$. $\mathscr{M}\mathscr{O}_{k,s}(M, \omega)$ is equipped with the topology induced by $\mathrm{Sym}^s \Sigma$ and $\mathscr{M}(M, \omega)$. •

By Proposition 2.8.5, the failure of super-rigidity is detected by the following subsets.

Definition 2.9.2. Let $s \in \mathbf{N}_0$. Denote by $\mathscr{W}_{\blacklozenge,s}(M, \omega)$ the subset of those $(J, [u, j; \nu]) \in \mathscr{M}\mathscr{O}_{0,s}(M, \omega)$ for which there exists an irreducible Euclidean local system $\underline{V}$ on Σ_ν such that:

(1) the monodromy representation of $\underline{V}$ factors through a finite quotient of $\pi_1(\Sigma, x_0)$,

(2) $\underline{V}$ has non-trivial monodromy around every $x \in Z_\nu$, and

(3) $\ker \mathfrak{d}_{u,J;\nu}^{N,\underline{V}} \neq 0$.

Denote by $\mathscr{W}_{\blacklozenge,s}^{\mathrm{top}}(M, \omega)$ the subset of those $(J, [u, j; \nu]) \in \mathscr{M}\mathscr{O}_{0,s}(M, \omega)$ for which there exists $\underline{V}$ with all of the above properties and satisfying additionally:

(4) $\dim \ker \mathfrak{d}_{u,J;\nu}^{N,\underline{V}} = 1$,

(5) if $\underline{V}'$ is any irreducible Euclidean local system on Σ_ν not isomorphic to $\underline{V}$ and whose monodromy representation factors through a finite quotient of $\pi_1(\Sigma, x_0)$, then

$$\dim \ker \mathfrak{d}_{u,J;\nu}^{N,\underline{V}'} = 0.$$

(6) if $n = 3$, then $\dim(V / V^{\rho_x}) = 1$ for every $x \in Z_\nu$; otherwise, $s = 0$. •

To prove that $\mathscr{W}_{\blacklozenge}(M, \omega)$ has codimension at least one, we will verify that condition (3.c) from Definition 2.2.6 holds; the other conditions are verified in the same way as in the proof of Theorem 2.4.2. To that end, consider, as in the proof of Proposition 2.6.1, a point $(J_0, [u_0, j_0]) \in \mathscr{M}_0(M, \omega)$ with u_0 being an embedding, and a sequence $\varepsilon \in \mathfrak{s}$ converging to zero. Let $\mathscr{V}$ be an open neighborhood of $(J_0, [u_0, j_0])$ in the universal moduli space of simple maps over $\mathscr{U} = \mathscr{U}(M, \omega; J_0, \varepsilon) \subset \mathscr{J}(M, \omega)$.

Notation 2.9.3. Given a subset $\mathscr{V} \subset \mathscr{M}_k(M,\omega)$, denote by $\mathscr{M}\mathscr{O}_{k,s}(\mathscr{V})$, $\mathscr{W}_{\blacklozenge,s}(\mathscr{V})$, and $\mathscr{W}^{\mathrm{top}}_{\blacklozenge,s}(\mathscr{V})$ the subsets of the corresponding spaces consisting of $(J,[u,j;\nu])$ such that $(J,[u,j]) \in \mathscr{V}$. ∘

In the situation at hand, we equip these spaces with the topology induced from the Banach space topology on $\mathscr{U}$. We can choose ε, $\mathscr{U}$, and $\mathscr{V}$ so that the following hold:

- $\mathscr{V}$ maps onto $\mathscr{U}$ under the projection $\mathscr{M}_0(M,\omega) \to \mathscr{J}(M,\omega)$.
- For every $(J,[u,j]) \in \mathscr{V}$ the map u is an embedding.
- $\mathscr{V}$ is liftable in the sense that there is an $\mathrm{Aut}(\Sigma, j_0)$–invariant slice $\mathscr{S}$ of Teichmüller space through j_0 such that for every $(J,[u,j]) \in \mathscr{V}$ there is a unique lift $u\colon (\Sigma,j) \to (M,J)$ with $j \in \mathscr{S}$.
- $\mathscr{M}\mathscr{O}_{0,s}(\mathscr{V})$ is a Banach manifold and the map $\Pi_{0,s}\colon \mathscr{M}\mathscr{O}_{0,s}(\mathscr{V}) \to \mathscr{U}$ is a Fredholm map of index $2s$. (The existence of ε and $\mathscr{V}$ satisfying this condition is proved in the same way as Theorem 2.4.2.)

Proposition 2.9.4. *For every $s \in \mathbf{N}_0$ there is a closed subset $\mathscr{W}^{\Lambda}_{\blacklozenge,s}(\mathscr{V}) \subset \mathscr{M}\mathscr{O}_{0,s}(\mathscr{V})$ of infinite codimension such that the following hold:*

(1) $\mathscr{W}^{\mathrm{top}}_{\blacklozenge,s}(\mathscr{V})\backslash\mathscr{W}^{\Lambda}_{\blacklozenge,s}(\mathscr{V})$ *is contained in a submanifold of codimension* $2s+1$.

(2) $\mathscr{W}_{\blacklozenge,s}(\mathscr{V})\backslash(\mathscr{W}^{\mathrm{top}}_{\blacklozenge,s}(\mathscr{V}) \cup \mathscr{W}^{\Lambda}_{\blacklozenge,s}(\mathscr{V}))$ *has codimension at least* $2s+2$.

Proof. As in the proof of Proposition 2.6.1, the operators $\mathfrak{d}^N_{u,J;\nu}\colon \Gamma((Nu)_\nu) \to \Omega^{0,1}(\Sigma_\nu,(Nu)_\nu)$ for $(J,[u,j,\nu]) \in \mathscr{V}$ can be regarded as a family of linear elliptic operators

$$\mathfrak{d}\colon \mathscr{V} \to \mathscr{F}(W^{1,2}\Gamma(E), L^2\Gamma(F))$$

as in Definition 1.1.1.

Let $\underline{V}_1, \underline{V}_2$ be a pair of non-isomorphic irreducible Euclidean local system whose monodromy representation factors through a finite quotient of $\pi_1(\Sigma,x_0)$ and consider $\mathfrak{V}$ as in Situation 1.3.1 consisting of $\underline{V}_1$ and $\underline{V}_2$. Denote by $\mathscr{W}^{\Lambda}_{\blacklozenge,s;\mathfrak{V}}(\mathscr{V})$ the subset of those $p \coloneqq (J,[u,j]) \in \mathscr{V}$ for which the map $\Lambda^{\mathfrak{V}}_p$ defined in Theorem 1.3.5 fails to be surjective. The argument from the proof of Proposition 2.6.1 shows that $\mathscr{W}^{\Lambda}_{\blacklozenge,s;\mathfrak{V}}(\mathscr{V})$ is a closed subset of infinite codimension. The union of these subsets is $\mathscr{W}^{\Lambda}_{\blacklozenge,s}(\mathscr{V})$. For $d \in \mathbf{N}_0^2$ set

$$\mathscr{W}^{d}_{\blacklozenge,s;\mathfrak{V}}(\mathscr{V}) \coloneqq \Big\{(J,[u,j]) \in \mathscr{U}\backslash\mathscr{W}^{\Lambda}_{\blacklozenge,s;\mathfrak{V}}(\mathscr{V}) : \dim_{\mathbf{K}_\alpha} \ker \mathfrak{d}^{N,\underline{V}_\alpha}_{u,J;\nu} = d_\alpha \text{ for } \alpha = 1,2\Big\}.$$

By Theorem 1.3.5, $\mathscr{W}^{d}_{\blacklozenge,s;\mathfrak{V}}(\mathscr{V})$ is a submanifold of codimension

$$\operatorname{codim} \mathscr{W}^{d}_{\blacklozenge,s;\mathfrak{V}}(\mathscr{V}) = \sum_{\alpha=1}^{2} k_\alpha d_\alpha(d_\alpha - i_\alpha) \quad \text{with} \quad i_\alpha \coloneqq \operatorname{index}_{\mathbf{K}_\alpha} \mathfrak{d}^{N,\underline{V}_\alpha}_{u,J;\nu}.$$

By Proposition 2.8.6,

$$i_\alpha \coloneqq \operatorname{index}_{\mathbf{K}_\alpha} \mathfrak{d}^{N,\underline{V}_\alpha}_{u,J;\nu} \leqslant -(n-1)\sum_{x\in Z_\nu} \dim_{\mathbf{K}_\alpha}(V_\alpha/V_\alpha^{\rho_x}).$$

If $\underline{V}_\alpha$ has non-trivial monodromy around every $x \in Z_\nu$, then $i_\alpha \leqslant -(n-1)r$. Therefore, if $d_\alpha \geqslant 1$, then

$$\operatorname{codim} \mathscr{W}^d_{\blacklozenge,s;\mathfrak{B}}(\mathscr{V}) \geqslant (n-1)s + 1 \geqslant 2s + 1.$$

$\mathscr{W}_{\blacklozenge,s}(\mathscr{V}) \backslash \mathscr{W}^\Lambda_{\blacklozenge,s}(\mathscr{V})$ is the union of countably many subsets of the form $\mathscr{W}^d_{\blacklozenge,s;\mathfrak{B}}(\mathscr{V})$ with at least one $\alpha = 1, 2$ as above. Therefore, it has codimension at least $2s+1$.

Analysing the chain of inequalities shows that $\operatorname{codim} \mathscr{W}^d_{\blacklozenge,s;\mathfrak{B}}(\mathscr{V}) = 2s+1$ if and only if there is an $\alpha = 1, 2$ such that:

(1) $d_\alpha = 1$, $\mathbf{K}_\alpha = \mathbf{R}$ and $d_\beta = 0$ for $\beta \neq \alpha$, and

(2) if $n = 3$, then $\dim(V_\alpha / V_\alpha^{\rho_x}) = 1$ for every $x \in Z_\nu$; otherwise, $Z_\nu = \varnothing$.

The union of these subsets is $\mathscr{W}^{\mathrm{top}}_{\blacklozenge,s}(\mathscr{V}) \backslash \mathscr{W}^\Lambda_{\blacklozenge,s}(\mathscr{V})$; hence: (1) holds. Furthermore, only $\mathscr{W}^d_{\blacklozenge,s;\mathfrak{B}}(\mathscr{V})$ of codimension at least $2s+2$ are required to cover $\mathscr{W}_{\blacklozenge,s}(\mathscr{V}) \backslash (\mathscr{W}^{\mathrm{top}}_{\blacklozenge,s}(\mathscr{V}) \cup \mathscr{W}^\Lambda_{\blacklozenge,s}(\mathscr{V}))$. This implies (2). ■

Proof of Theorem 2.1.13. The subset of $J \in \mathscr{J}(M, \omega)$ which fail to be super-rigid is

$$\mathscr{W}_{\geqslant 0}(M, \omega) \cup \mathscr{W}_{\hookrightarrow}(M, \omega) \cup \mathscr{W}_{\blacklozenge}(M, \omega).$$

The first two subsets have already been shown to have codimension at least two. To show that $\mathscr{W}_\blacklozenge(M, \omega)$ has codimension at least one, observe that, the intersection of $\mathscr{W}_\blacklozenge(M, \omega)$ with $\mathscr{U} = \mathscr{U}(M, \omega; J_0, \varepsilon)$ is contained in

$$\bigcup_{s \in \mathbf{N}_0} \Pi_{0,s}(\mathscr{W}_{\blacklozenge,s}(\mathscr{V})).$$

This verifies condition (3.c) from Definition 2.2.6 since $\Pi_{s,0}$ has index $2s$ and, by Proposition 2.9.4, $\mathscr{W}_{\blacklozenge,s}(\mathscr{V})$ has codimension at least $2s+1$ in $\mathscr{U}$. ■

2.10 Super-rigidity along paths of almost complex structures

The following describes in detail how super-rigidity may fail along a generic path of almost complex structures. In what follows, it is convenient to use the **fibered product** notation. Given continuous maps of topological spaces $f\colon X \to Z$ and $g\colon Y \to Z$, the fibered product of X and Y along Z is

$$X \times_Z Y = \{(x, y) \in X \times Y : f(x) = g(y)\}.$$

Definition 2.10.1. Let $\boldsymbol{\mathscr{J}}(M, \omega)$ be the space of paths $\mathbf{J}\colon [0,1] \to \mathscr{J}(M, \omega)$ which are smooth as a section over $[0,1] \times M$. Equip $\boldsymbol{\mathscr{J}}(M, \omega)$ with the C^∞ topology.

For $\mathbf{J} \in \boldsymbol{\mathscr{J}}(M, \omega)$ set $J_t \coloneqq \mathbf{J}(t)$. Denote by $\boldsymbol{\mathscr{J}}_\blacklozenge(M, \omega)$ the subset of all $\mathbf{J} \in \boldsymbol{\mathscr{J}}(M, \omega)$ for which the following conditions hold:

(1) The 1–parameter moduli space of pseudo-holomorphic maps

$$\boldsymbol{\mathscr{M}}_0(M, \mathbf{J}) \coloneqq [0,1] \times_{\mathscr{J}(M,\omega)} \mathscr{M}_0(M, \omega),$$

is a 1–dimensional manifold with boundary. Here the fibered product is taken with respect to the path $\mathbf{J}\colon [0,1] \to \mathscr{J}(M, \omega)$ and the projection $\mathscr{M}_0(M, \omega) \to \mathscr{J}(M, \omega)$.

(2) For every $t \in [0,1]$ the following hold:

   (a) Every simple J_t–holomorphic map has non-negative index.

   (b) Every simple J_t–holomorphic map of index zero is an embedding, and every two simple J_t–holomorphic maps of index zero either have disjoint images or are related by a reparametrization.

(3) The set $I_\blacklozenge$ of those $t \in [0,1]$ for which J_t fails to be super-rigid is countable; moreover, $t \in I_\blacklozenge$ if and only if

$$J_t \in \bigcup_{s\in\mathbf{N}_0} \Pi_{0,s}(\mathscr{W}^{\mathrm{top}}_{\blacklozenge,s}(M,\omega)). \qquad \bullet$$

Theorem 2.10.2 (cf. [Wen19b, Section 2.4]). *$\mathscr{J}_\blacklozenge(M,\omega)$ is a comeager subset of $\mathscr{J}(M,\omega)$.*

Proof of Theorem 2.10.2. The set of $\mathbf{J}$ satisfying conditions (1) and (2) from Definition 2.10.1 is comeager by the same arguments as in the proofs of Theorem 2.4.2 and Theorem 2.4.3; this is standard.

By Proposition 2.2.7, to prove that condition (3) from Definition 2.10.1 is satisfied for $\mathbf{J}$ from a comeager set it suffices to consider the following local situation. Fix $\mathbf{J}_0$, $\varepsilon \in \mathfrak{s}$, and $t_0 \in [0,1]$, and let $\mathscr{U}(M,\omega;\mathbf{J}_0;\varepsilon)$ be the C^ε neighborhood of $\mathbf{J}_0$ in $\mathscr{J}(M,\omega)$; here we think of elements of $\mathscr{J}(M,\omega)$ as sections over $[0,1]\times M$ and define the C^ε topology in the same way as in Definition 2.2.3 and Definition 2.2.5, with M replaced by $[0,1]\times M$. Let $\mathscr{V}$ be an open subset of the universal moduli space of simple index zero maps over $\mathscr{U}(M,\omega,\mathbf{J}_0(t_0),\varepsilon)$, with the properties listed in Section 2.9. For a sufficiently small open neighborhood $I \subset [0,1]$ of t_0, there is a well-defined evaluation map

$$\begin{aligned} \mathrm{ev}\colon\ I\times\mathscr{U}(M,\omega;\mathbf{J}_0;\varepsilon) &\to \mathscr{U}(M,\omega;\mathbf{J}_0(t_0);\varepsilon) \\ \mathrm{ev}(t,\mathbf{J}) &\coloneqq \mathbf{J}(t), \end{aligned}$$

which is a submersion of Banach manifolds. By Proposition 2.9.4, the preimage of

$$\bigcup_{s\in\mathbf{N}_0} \Pi_{0,s}(\mathscr{W}_{\blacklozenge,s}(\mathscr{V})\backslash(\mathscr{W}^{\mathrm{top}}_{\blacklozenge,s}(\mathscr{V})\cup\mathscr{W}^{\Lambda}_{\blacklozenge,s}(\mathscr{V})))$$

under the evaluation map has codimension at least two in $I\times\mathscr{U}(M,\omega;\mathbf{J}_0;\varepsilon)$. Therefore, its image in $\mathscr{U}(M,\omega;\mathbf{J}_0;\varepsilon)$ has codimension at least one; that is: a generic path in $\mathscr{U}(M,\omega;\mathbf{J}_0;\varepsilon)$ either avoids the subsets $\Pi_{0,s}(\mathscr{W}_{\blacklozenge,s}(\mathscr{V}))$ or intersects them at one of the subsets $\Pi_{0,s}(\mathscr{W}^{\mathrm{top}}_{\blacklozenge,s}(\mathscr{V}))$. The set of points $t\in I$ at which the latter happens is codimension one in I, and therefore countable. ■

2.A The normal Cauchy–Riemann operator

The normal Cauchy–Riemann operator for embedded J–holomorphic maps can be traced back to the work of Gromov [Gro85, 2.1.B]. It was observed by Ivashkovich and Shevchishin [IS99, Section 1.3] that the normal Cauchy–Riemann operator can be defined even for non-embedded

J–holomorphic maps, and that it plays an important role in understanding the deformation theory of J–holomorphic curves; see also [Wen10, Section 3]. In this section we will briefly explain the construction of Tu and Nu, and discuss the proof of Proposition 2.1.7.

Definition 2.A.1. Let $u\colon (\Sigma, j) \to (M, J)$ be a non-constant J–holomorphic map. Denote by $\mathfrak{d}_{u,J}$ the linearization of the J–holomorphic map equation introduced in (2.1.5). Denote by $\bar\partial_{u,J}$ the complex linear part of $\mathfrak{d}_{u,J}$. This is a complex Cauchy–Riemann operator and gives u^*TM the structure of a holomorphic vector bundle

$$\mathscr{E} := (u^*TM, \bar\partial_{u,J}).$$

Denote by $\mathscr{T}\Sigma$ the tangent bundle of Σ equipped with its natural holomorphic structure. The derivative of u induces a holomorphic map $\mathrm{d}u\colon \mathscr{T}\Sigma \to \mathscr{E}$. The quotient of this map, thought of as a morphism of sheaves,

$$\mathscr{Q} := \mathscr{E}/\mathscr{T}\Sigma$$

is a coherent sheaf on Σ. Denote by $\mathrm{Tor}(\mathscr{Q})$ the torsion subsheaf of $\mathscr{Q}$. The quotient

$$\mathscr{N}u := \mathscr{Q}/\mathrm{Tor}(\mathscr{Q})$$

is torsion-free; hence: locally free. The corresponding holomorphic vector bundle $(Nu, \bar\partial_{Nu})$ is called the **generalized normal bundle of** u. The kernel

$$\mathscr{T}u := \ker(\mathscr{E} \to \mathscr{N}u).$$

also is locally free. The corresponding holomorphic vector bundle $(Tu, \bar\partial_{Tu})$ is called the **generalized tangent bundle of** u. •

Proposition 2.A.2. *Denote by D the divisor of critical points of* $\mathrm{d}u$ *counted with multiplicty. There is a short exact sequence*

$$0 \to \mathscr{T}\Sigma \to \mathscr{T}u \to \mathscr{O}_D \to 0;$$

in particular:

$$\mathscr{T}u \cong \mathscr{T}\Sigma(D).$$

Proof. The following commutative diagram summarizes the construction of $\mathscr{T}u$ and $\mathscr{N}u$:

$$\begin{tikzcd}
 & & \mathrm{Tor}(\mathscr{Q}) \arrow[d, hook] \\
\mathscr{T}\Sigma \arrow[r, hook] \arrow[d, hook] & \mathscr{E} \arrow[r, two heads] \arrow[d, equal] & \mathscr{Q} \arrow[d, two heads] \\
\mathscr{T}u \arrow[r, hook] \arrow[d, two heads] & \mathscr{E} \arrow[r, two heads] & \mathscr{N}u \\
\mathscr{T}u/\mathscr{T}\Sigma. & &
\end{tikzcd}$$

Since the columns and rows are exact sequences, it follows from the Snake Lemma that

$$\mathrm{Tor}(\mathcal{Q}) \cong \mathcal{T}u/\mathcal{T}\Sigma.$$

Thus it remains to prove that $\mathrm{Tor}(\mathcal{Q}) \cong \mathcal{O}_D$. This is a consequence of the fact that near a critical point z_0 of order k we can write $\mathrm{d}u$ as $(z - z_0)^k f(z)$ with $f(z_0) \neq 0$. ∎

Proposition 2.A.3. *Let $u\colon (\Sigma, j) \to (M, J)$ be a non-constant J–holomorphic map. If $\pi\colon (\tilde{\Sigma}, \tilde{j}) \to (\Sigma, j)$ is a non-constant holomorphic map and $\tilde{u} := u \circ \pi$, then*

$$\mathcal{T}\tilde{u} \cong \pi^*\mathcal{T}u \quad \textit{and} \quad \mathcal{N}\tilde{u} \cong \pi^*\mathcal{N}u.$$

The corresponding isomorphism of vector bundles $N\tilde{u} \cong \pi^* Nu$ induces a commutative diagram

$$\begin{tikzcd}
\Gamma(\tilde{\Sigma}, N\tilde{u}) \arrow[r, "\mathfrak{d}^N_{\tilde{u},J}"] \arrow[d, "\cong"] & \Omega^{0,1}(\tilde{\Sigma}, N\tilde{u}) \arrow[d, "\cong"] \\
\Gamma(\tilde{\Sigma}, \pi^* Nu) \arrow[r, "\pi^\star \mathfrak{d}^N_{u,J}"] & \Omega^{0,1}(\tilde{\Sigma}, \pi^* Nu).
\end{tikzcd}$$

Proof. $\mathcal{T}u \subset \mathcal{E}$ is the minimal locally free subsheaf which contains the image of $\mathcal{T}\Sigma \hookrightarrow \mathcal{E}$. Set $\tilde{\mathcal{E}} := (\tilde{u}^*TM, \bar{\partial}_{\tilde{u},j})$. There is a canonical isomorphism $\tilde{\mathcal{E}} \cong \pi^*\mathcal{E}$. Through this identification, $\pi^*\mathcal{T}u$ can be regarded as a subsheaf of $\tilde{\mathcal{E}}$. It is locally free and contains the image of $\mathcal{T}\tilde{\Sigma} \hookrightarrow \tilde{\mathcal{E}}$. Therefore, $\mathcal{T}\tilde{u} \cong \pi^*\mathcal{T}u$. This also implies that $\mathcal{N}\tilde{u} \cong \tilde{\mathcal{E}}/\mathcal{T}\tilde{u} \cong \pi^*(\mathcal{E}/\mathcal{T}u) \cong \pi^*\mathcal{N}u$.

That the isomorphism $N\tilde{u} \cong \pi^* Nu$ identifies $\pi^*\mathfrak{d}^N_{u,J}$ and $\mathfrak{d}^N_{\tilde{u},J}$ is evident away from the set of critical points of π. Since the latter is nowhere dense, the operators are identified everywhere. ∎

Proof of Proposition 2.1.7. Let $\mathcal{S}$ be an $\mathrm{Aut}(\Sigma, j)$–invariant local slice of the Teichmüller space $\mathcal{T}(\Sigma)$ through j. Recall that $\mathrm{d}_{u,j}\bar{\partial}_J\colon \Gamma(u^*TM) \oplus T_j\mathcal{S} \to \Omega^{0,1}(\Sigma, u^*TM)$ is the linearization of $\bar{\partial}_J$, defined in (2.1.2), restricted to $C^\infty(\Sigma, M) \times \mathcal{S}$. Denote by Tu the complex vector bundle underlying $\mathcal{T}u$ and by Nu the complex vector bundle underlying $\mathcal{N}u$. As was mentioned before Definition 2.1.6, $Tu \subset u^*TM$ is the unique complex subbundle of rank one containing $\mathrm{d}u(T\Sigma)$. Using a Hermitian metric on u^*TM we obtain an isomorphism

$$u^*TM \cong Tu \oplus Nu.$$

With respect to this splitting $\mathfrak{d}_{J,u}$, the restriction of $\mathrm{d}_{u,j}\bar{\partial}_J$ to $\Gamma(u^*TM)$, can be written as

$$\mathfrak{d}_{J,u} = \begin{pmatrix} \mathfrak{d}^T_{u,J} & * \\ \dagger & \mathfrak{d}^N_{u,J} \end{pmatrix}$$

with $\mathfrak{d}^N_{u,J}$ denoting the normal Cauchy–Riemann operator introduced in Definition 2.1.6. Since

$$\bar{\partial}_{u,J} \circ \mathrm{d}u = \mathrm{d}u \circ \bar{\partial}_{T\Sigma} \quad \text{and} \quad \mathcal{T}u \cong \mathcal{T}\Sigma(D),$$

it follows that

$$\mathfrak{d}^T_{u,J} = \bar{\partial}_{Tu} \quad \text{and} \quad \dagger = 0.$$

Denote by $\iota\colon T_j\mathcal{S} \to \Omega^{0,1}(\Sigma, u^*TM)$ the restriction of $\mathrm{d}_{u,j}\bar\partial_J$ to $T_j\mathcal{S}$. The tangent space to the Teichmüller space $\mathcal{T}(\Sigma)$ at $[j]$ can be identified with $\operatorname{coker}\bar\partial_{T\Sigma} \cong \ker\bar\partial_{T\Sigma}^*$. With respect to this identification, ι is the restriction of $\mathrm{d}u\colon T\Sigma \to u^*TM$ to $\ker\bar\partial_{T\Sigma}^*$. Consequently, we can write $\mathrm{d}_{u,j}\bar\partial_J\colon \Gamma(Tu) \oplus T_j\mathcal{S} \oplus \Gamma(Nu) \to \Gamma(Tu) \oplus \Gamma(Nu)$ as

$$\mathrm{d}_{u,j}\bar\partial_J = \begin{pmatrix} \bar\partial_{Tu} & \iota & * \\ 0 & 0 & \mathfrak{d}_{u,J}^N \end{pmatrix}.$$

The short exact sequence

$$0 \to \mathcal{T}\Sigma \to \mathcal{T}u \to \mathcal{O}_D \to 0$$

induces the following long exact sequence in cohomology

$$0 \to H^0(\mathcal{T}\Sigma) \to H^0(\mathcal{T}u) \to H^0(\mathcal{O}_D) \to H^1(\mathcal{T}\Sigma) \to H^1(\mathcal{T}u) \to 0.$$

It follows that

$$\operatorname{index}\bar\partial_{Tu} = 2\chi(\mathcal{T}u) = 2\chi(\mathcal{T}\Sigma) + 2h^0(\mathcal{O}_D) = \operatorname{index}\bar\partial_{T\Sigma} + 2Z(\mathrm{d}u),$$

and, moreover, that $\ker\bar\partial_{T\Sigma} \to \ker\bar\partial_{Tu}$ is injective, and $\operatorname{coker}\bar\partial_{T\Sigma} \to \operatorname{coker}\bar\partial_{Tu}$ is surjective. The latter implies that $\bar\partial_{Tu} \oplus \iota$ is surjective. Therefore, there is an exact sequence

$$0 \to \ker\bar\partial_{Tu} \oplus \iota \to \ker\mathrm{d}_{u,j}\bar\partial_J \to \ker\mathfrak{d}_{u,J}^N \to 0,$$

and an isomorphism

$$\operatorname{coker}\mathrm{d}_{u,j}\bar\partial_J \cong \operatorname{coker}\mathfrak{d}_{u,J}^N.$$

The kernel of $\bar\partial_{Tu} \oplus \iota$ contains $\mathfrak{aut}(\Sigma, j) = \ker\bar\partial_{T\Sigma}$ and

$$\begin{aligned} \dim\ker\bar\partial_{Tu} \oplus \iota &= \operatorname{index}\bar\partial_{Tu} \oplus \iota \\ &= \operatorname{index}\bar\partial_{Tu} + \dim T_j\mathcal{S} \\ &= \operatorname{index}\bar\partial_{T\Sigma} + \dim T_j\mathcal{S} + 2Z(\mathrm{d}u) \\ &= \dim\mathfrak{aut}(\Sigma, j) + 2Z(\mathrm{d}u). \end{aligned}$$

This completes the proof of Proposition 2.1.7. ■

2.B Orbifold Riemann–Roch formula

The purpose of this section is to prove Proposition 2.8.6. The proof relies on Kawasaki's orbifold Riemann–Roch theorem [Kaw79] and a result due to Ohtsuki [Oht82]. The Riemann–Roch theorem for complex orbifolds is not easy to digest; however, for orbifold Riemann surfaces it simplifies significantly and can be proved by an elementary argument based on the discussion in [FS92, Section 1; NS95, Section 1B; KM95, Sections 8(ii)–(iii)].

This argument relies on the following local considerations. Let $\rho\colon \mu_k \to \mathrm{GL}(V)$ be a representation and let μ_k act on $\mathbf{D} \times V$ via $\zeta \cdot (z, v) := (\zeta z, \rho(\zeta)v)$.

$$V_\rho := [(\mathbf{D} \times V)/\mu_k]$$

is a vector bundle over $[\mathbf{D}/\mu_k]$. In fact, up to isomorphism, every vector bundle over $[\mathbf{D}/\mu_k]$ is of this form. V_ρ and V_σ are isomorphic if and only if the representations ρ and σ are. However, if ρ is a complex representation, then the restriction V_ρ to $\dot{\mathbf{D}} := [(\mathbf{D}\backslash\{0\})/\mu_k]$ is trivial. This is a consequence of the fact that $\mathrm{GL}_r(\mathbf{C})$ is connected; more concretely, it can be seen as follows. Choose an isomorphism $V \cong \mathbf{C}^r$ with respect to which ρ is diagonal; that is:

$$\rho(\zeta) = \begin{pmatrix} \zeta^{w_1} & & \\ & \ddots & \\ & & \zeta^{w_r} \end{pmatrix} \tag{2.B.1}$$

for $w \in (\mathbf{Z}/k\mathbf{Z})^r$. A choice of lift of w to $\tilde{w} \in \mathbf{Z}^r$ extends ρ to a representation $\hat{\rho}\colon\ \mathbf{C}^* \to \mathrm{GL}(V)$. The (inverse of the) map $\eta\colon\ V_\rho|_{\dot{\mathbf{D}}} \to \dot{\mathbf{D}} \times V$ defined by

$$\eta([z, v]) := [z, \hat{\rho}(z^{-1})v] \tag{2.B.2}$$

trivializes V_ρ over $\dot{\mathbf{D}}$. The trivial bundle over $[\mathbf{D}/\mu_k]$ with fiber V and the bundle $V_\rho \to [\mathbf{D}/\mu_k]$ have canonical holomorphic structures. Denote by $\mathscr{V}$ and $\mathscr{V}_\rho$ the corresponding sheaves of holomorphic orbifold sections. The map η is holomorphic with respect to the canonical holomorphic structures. If $\tilde{w}$ is chosen in $(-k, 0]^r$, then η induces a sheaf morphism $\eta\colon\ \mathscr{V}_\rho \to \mathscr{V}$. To see this, observe that if s is a germ of a section of $\mathscr{V}_\rho$ at $[0]$, then $\eta(s)$ is bounded and thus defines a germ of a section of $\mathscr{V}$ at $[0]$. Evidently, η is injective. Furthermore, it fits into the exact sequence

$$\mathscr{V}_\rho \hookrightarrow \mathscr{V} \twoheadrightarrow V/V^\rho \otimes \mathscr{O}_0. \tag{2.B.3}$$

Here $\mathscr{O}_0$ denotes the structure sheaf of the point $[0]$. To see this it suffices to consider the case $r = 1$. A germ of a section of $\mathscr{V}^\rho$ at $[0]$ is nothing but a germ of a holomorphic map $s\colon\ \mathbf{D} \to \mathbf{C}$ which is μ_k–equivariant; that is: $s(\zeta z) = \zeta^w s(z)$ for every $\zeta \in \mu_k$. The map η is given by $(\eta s)(z) := z^{-\tilde{w}} s(z)$. If $w = 0$, then η is the identity and the final map in (2.B.3) is trivial. If $w \neq 0$, then the final map in (2.B.3) is the evaluation map at 0. The Taylor expansion of a germ of a μ_k–invariant holomorphic map $t\colon\ \mathbf{D} \to \mathbf{C}$ involves powers of z^k. Therefore, if t vanishes at 0, then $s := z^{\tilde{w}} t$ is a germ of a μ_k–equivariant holomorphic map such that $\eta s = t$. (Here it is crucial that $\tilde{w} \geqslant -k$.)

Definition 2.B.4. Let (Σ, j) be a Riemann surface with a multiplicity function ν and let $\mathscr{E} = (E, \bar{\partial})$ be a holomorphic vector bundle over Σ. Define (Σ_ν, j_ν) and $\beta_\nu\colon\ (\Sigma_\nu, j_\nu) \to (\Sigma, j)$ as in Definition 2.7.1 and set $\mathscr{E}_\nu := \beta_\nu^* \mathscr{E}$. Let $\rho = (\rho_x\colon\ \mu_{\nu(x)} \to \mathrm{GL}(E_x))_{x \in Z_\nu}$ be a collection of representations. A **Hecke modification of $\mathscr{E}_\nu$ of type ρ** consists of a holomorphic vector bundle $\mathscr{E}_{\nu,\rho}$ over Σ_ν together with a holomorphic map

$$\eta\colon\ \mathscr{E}_{\nu,\rho}|_{\Sigma_\nu \backslash Z_\nu} \to \mathscr{E}_\nu|_{\Sigma_\nu \backslash Z_\nu}$$

such that for every $x \in Z_\nu$ with respect to suitable holomorphic trivializations of $\mathscr{E}_{\nu,\rho}$ and $\mathscr{E}_\nu$ around x the map η is of the form (2.B.2) with $\rho = \rho_x$ and $\tilde{w} \in (-k, 0]^r$. •

Remark 2.B.5. It is evident from the preceding discussion that every holomorphic vector bundle on (Σ_ν, j_ν) can be obtained by a Hecke modification. ♣

Theorem 2.B.6 (Orbifold Riemann–Roch Formula [Kaw79]). *In the situation of Definition 2.B.4,*

$$\chi(\mathscr{E}_{\nu,\rho}) = \chi(\mathscr{E}_\nu) - \sum_{x\in Z_\nu} \dim_{\mathbf{C}}(E_x/E_x^{\rho_x}).$$

Proof. The exact sequence (2.B.3) induces the exact sequence

$$\mathscr{E}_{\nu,\rho} \hookrightarrow \mathscr{E}_\nu \twoheadrightarrow \bigoplus_{x\in Z_\nu} (E_x/E_x^{\rho_x}) \otimes \mathscr{O}_x.$$

This immediately implies the assertion. ■

The proof of Proposition 2.8.6 requires one more piece of preparation. In the situation of Definition 2.B.4, if $\mathscr{E}_{\nu,\rho}$ carries a holomorphic flat connection $\nabla_{\nu,\rho}$, then it induces a meromorphic flat connection ∇_ν on $\mathscr{E}$ with simple poles. With respect to suitable local holomorphic coordinates and trivializations around x

$$\nabla_\nu = \mathrm{d} + \mathrm{Res}_x(\nabla_\nu)\frac{\mathrm{d}z}{z} \quad \text{with} \quad \mathrm{Res}_x(\nabla_\nu) \coloneqq \frac{1}{\nu(x)}\begin{pmatrix} \tilde{w}_1(x) & & \\ & \ddots & \\ & & \tilde{w}_r(x)\end{pmatrix}.$$

Here $\tilde{w}_i(x)$ are as in the discussion preceding Definition 2.B.4. By (a very special case of) [Oht82, Theorem 3], the degree of E and the residues $\mathrm{Res}_x(\nabla_\nu)$ are related by

$$\deg E = -\sum_{x\in Z_\nu} \mathrm{tr}\,\mathrm{Res}_x(\nabla) = -\sum_{x\in Z_\nu}\sum_{i=1}^r \frac{\tilde{w}_i(x)}{\nu(x)}. \tag{2.B.7}$$

Proof of Proposition 2.8.6. Set $V^{\mathbf{C}} \coloneqq V \otimes \mathbf{C}$ and $\underline{V}^{\mathbf{C}} \coloneqq \underline{V} \otimes \underline{\mathbf{C}}$. For every $x \in Z_\nu$ denote by $\rho_x^{\mathbf{C}}\colon \mu_{\nu(x)} \to \mathrm{GL}_{\mathbf{C}}(V^{\mathbf{C}})$ the complexification of the monodromy representation of $\underline{V}$ around x. There is a holomorphic vector bundle $\mathscr{V}$ over Σ such that $\underline{V}^{\mathbf{C}} \cong \mathscr{V}_{\nu,\rho^{\mathbf{C}}}$. Equip E with the holomorphic structure $\bar\partial$ satisfying $\mathfrak{d} = \bar\partial + \mathfrak{n}$ with $\mathfrak{n} \in \Omega^{0,1}(\Sigma, \overline{\mathrm{End}}_{\mathbf{C}}(E))$.

By Theorem 2.B.6 and the classical Riemann–Roch formula,

$$\begin{aligned}\mathrm{index}\,\mathfrak{d}_\nu^{\underline{V}} &= 2\chi(\mathscr{E}_\nu \otimes_{\mathbf{C}} \underline{V}^{\mathbf{C}})\\ &= 2\chi(\mathscr{E}_\nu \otimes_{\mathbf{C}} \mathscr{V}) - 2\,\mathrm{rk}_{\mathbf{C}}\, E \sum_{z\in Z_\nu} \dim(V/V^{\rho_x})\\ &= \dim V\,\mathrm{index}\,\mathfrak{d} + 2\,\mathrm{rk}_{\mathbf{C}}\, E\left(\deg\mathscr{V} - \sum_{z\in Z_\nu}\dim(V/V^{\rho_x})\right).\end{aligned}$$

Therefore, it remains prove that

$$\deg \mathscr{V} = \frac{1}{2}\sum_{z\in Z_\nu} \dim(V/V^{\rho_x}).$$

Let $k \in \mathbf{N}$. The complexification of the trivial representation $\rho_0\colon \mu_k \to \mathrm{GL}(\mathbf{R})$ is the trivial representation $\rho_0^{\mathbf{C}}\colon \mu_k \to \mathrm{GL}_{\mathbf{C}}(\mathbf{C})$. Therefore, the corresponding weight in $(-k, 0]$ is 0. For

$w \in \mathbf{Z}/k\mathbf{Z}$ the complexification of the representation $\rho_w\colon \mu_k \to \mathrm{GL}(\mathbf{C})$ defined by $\rho_w(\zeta) \mapsto \zeta^w$ is the representation $\rho_w^{\mathbf{C}}\colon \mu_k \to \mathrm{GL}_{\mathbf{C}}(\mathbf{C}^2)$ defined by

$$\rho^{\mathbf{C}}(\zeta) := \begin{pmatrix} \zeta^w & \\ & \zeta^{-w} \end{pmatrix}.$$

Therefore, the corresponding weights in $(-k, 0]$ are of the form $\tilde{w}$ and $-(\tilde{w} + k)$. It follows from this discussion that for every representation $\mu_k \to \mathrm{GL}(V)$ the sum of the weights of the complexification is $-\frac{k}{2}\dim(V/V^\rho)$. This combined with (2.B.7) proves the desired identity for $\deg \mathscr{V}$. ∎

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