Dirac operators twisted by ramified Euclidean line bundles

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Abstract

This article is concerned with the analysis of Dirac operators D twisted by ramified Euclidean line bundles $(Z, \mathfrak{l})$—motivated by their relation with $\mathbf{Z}/2\mathbf{Z}$ harmonic spinors, which have appeared in various context in gauge theory and calibrated geometry. The closed extensions of D are described in terms of the Gelfand–Robbin quotient $\check{\mathbf{H}}$. Assuming that the branching locus Z is a closed cooriented codimension two submanifold, a geometric realisation of $\check{\mathbf{H}}$ is constructed. This, in turn, leads to an L^2 regularity theory.

Contents

1 Introduction

Let (X, g) be a closed Riemannian manifold of dimension n.

Definition 1.1 (cf. [LM89, Chapter II Definition 5.2; Bis89, §1(b)]). A **Dirac bundle with skew torsion** on (X, g) consists of:

(1) a Euclidean vector bundle S over X equipped with a skew-adjoint Clifford multiplication $\gamma\colon TX \to \mathfrak{o}(S)$; that is:
$$\gamma(v)^2 = -|v|^2\,\mathbf{1}_S$$
for every $v \in TX$; and

(2) an orthogonal covariant derivative ∇ on S and a 3–form $\mathrm{Tor} \in \Omega^3(X)$ such that γ is parallel with respect to ∇ and the orthogonal affine connection ∇^T on (X, g) defined by
$$\langle \nabla^T_u v, w\rangle = \langle \nabla^{\mathrm{LC}}_u v, w\rangle + \tfrac{1}{2}\,\mathrm{Tor}(u, v, w).$$
Here ∇^{LC} denotes the Levi-Civita connection of (X, g). •

Definition 1.2. A **ramified Euclidean line bundle** over X consists of:

(1) a closed subset $Z \subset X$, the **branching locus**, and

(2) a Euclidean line bundle $\mathfrak{l}$ over $X\backslash Z$

such that

(3) if $W \subset Z$ is closed and $\mathfrak{l}$ extends over $X\backslash W$, then $W = Z$. •

This article is concerned with the analysis of the Dirac operator associated with a Dirac bundle with skew torsion $(S, \gamma, \nabla, \mathrm{Tor})$ twisted by a ramified Euclidean line bundle $\mathfrak{l}$
$$D\colon\ H^1\Gamma(X\backslash Z, S \otimes \mathfrak{l}) \to L^2\Gamma(X\backslash Z, S \otimes \mathfrak{l})$$
and its extensions.

The authors' motivation for this stems from the following. Taubes has observed that the failure of compactness for a wide variety of generalised Seiberg–Witten equations—e.g.: stable flat $\mathrm{PSL}_2(\mathbf{C})$–connections in dimension three [Tau13a], anti-self-dual $\mathrm{SL}_2(\mathbf{C})$–connections in dimension four [Tau13b], the Seiberg–Witten equation with multiple spinors [Tau16], the Vafa–Witten equation [Tau17], and the Kapustin–Witten equation [Tau22]— leaves behind evidence in the form of a **$\mathbf{Z}/2\mathbf{Z}$ harmonic spinor**. The latter is a pair $(Z, \mathfrak{l}; \Phi)$ consisting of a ramified Euclidean line bundle $(Z, \mathfrak{l})$ and a harmonic spinor $\Phi \in \ker D$. $\mathbf{Z}/2\mathbf{Z}$ harmonic spinors also appear in Donaldson's work on adiabatic limits of coassociative Kovalev–Lefschetz fibrations of G_2–manifolds [Don17] and He's work on branched double covers of special Lagrangian submanifolds [He22].

In light of this, it is important to understand the universal moduli space of $\mathbf{Z}/2\mathbf{Z}$ harmonic spinors (allowing for g, γ, and ∇ to vary). The fundamental issue is that D is only left semi-Fredholm (under mild assumptions; see Hypothesis 2.1), but not Fredholm—except in edge cases, e.g., if $Z = \varnothing$ or $n = 2$. The naive expectation is that the ∞–dimensional cokernel of D can be compensated by wiggling the branching locus Z. In his PhD thesis, Takahashi [Tak15; Tak17] has made some initial progress in this direction. Donaldson [Don21] and Parker [Par23] have developed a (partial) deformation theory for $\mathbf{Z}/2\mathbf{Z}$ harmonic 1–forms and $\mathbf{Z}/2\mathbf{Z}$ harmonic spinors on spin 3–manifolds respectively. There is work in progress by He, Parker and Walpuski to address this problem a bit more systematically. The present article should be considered infrastructure for this project (and, hopefully, other applications as well).

Here is a summary of the results contained in this article. Section 2 considers D as an unbounded operator $D_{\min}$ on $L^2\Gamma(X\backslash Z, S \otimes \mathfrak{l})$, the **minimal extension**, and systematically studies its closed extensions. The adjoint $D_{\max} \coloneqq D_{\min}^*$ is the **maximal extension** of $D_{\min}$. The closed extensions D_B of $D_{\min}$ are classified by **residue conditions**, that is: closed subspaces $B \subset \check{\mathbf{H}}$ of the **Gelfand–Robbin quotient**

$$\check{\mathbf{H}} \coloneqq \frac{\operatorname{dom}(D_{\max})}{\operatorname{dom}(D_{\min})}.$$

Moreover, $\check{\mathbf{H}}$ is equipped with a symplectic form G, the **Green's form**, which controls the formation of adjoints. Within this framework it is also possible to describe which extensions D_B are Fredholm. The entire discussion only relies on $D_{\min}$ being closed, densely defined, and symmetric as well as left semi-Fredholm. It is confined to the realm of abstract functional analysis and its purpose is to separate what is true for formal reasons from what is true for geometric reasons. Most of the observations in Section 2 can be found in [MS98, Exercise 2.17; BF98, §3; SW08, Appendix B; BS18, Exercises 6.3.3 and 6.5.11] in some shape or form.

Assuming that $Z \subset X$ is a closed (cooriented) submanifold of codimension two, Section 3 constructs an isomorphism of symplectic Hilbert spaces

$$\operatorname{res}\colon\ (\check{\mathbf{H}}, G) \cong (\check{H}\Gamma(Z, \check{S}), \check{\Omega}),$$

the **residue map**, between the Gelfand–Robbin quotient and a Hilbert space of sections of a symplectic vector bundle over Z. The residue map extracts the leading order behavior of $\phi \in \operatorname{dom}(D_{\max})$ which is shown to be (at worst) comparable to $\bar{z}^{-1/2}$ transversely to Z. With the help of the above it is possible to define spectral residue conditions, analogous to the APS boundary condition [APS75], as well as local residue conditions. As by product this yields a variant of the bordism theorem, whose significance remains somewhat mysterious to the authors. Evidently the construction in Section 3 is inspired by Bär and Ballmann's magnificent article [BB12] on boundary value problems for Dirac operators.

Finally, Section 4 develops an L^2 regularity theory on the scale of **adapated Sobolev spaces** $(H_a^k\Gamma(X\backslash Z, S \otimes \mathfrak{l}))_{k\in\mathbf{N}_0}$. This scale is defined via the ring of differential operators generated by conormal differential operators and the Dirac operator D. It gives rise to a graded Fréchet space

$H_a^\infty\Gamma(X\backslash Z, S\otimes \mathfrak{l})$ which is tame in the sense of Hamilton [Ham82, Part II Definition 1.3.2]—a prerequisite for using Nash–Moser theory. Moreover, spinors $\phi \in H_a^\infty\Gamma(X\backslash Z, S\otimes \mathfrak{l})$ extend smoothly across Z after untwisting by $\bar{z}^{1/2}$ and, therefore, have well-behaved polyhomogeneous expansions near Z. Crucially, it is proved that if a residue condition $B \subset \check{H}\Gamma(Z, \check{S})$ is ∞**–regular**, then the extension D_B satisfies a variant of elliptic regularity together with elliptic estimates. For local residue conditions B, ∞–regularity can be verified using straight-forward symbolic criterion. In particular, this criterion applies to the Lagrangian local residue condition which is secretly at the heart of [Tak15; Par23]. It is quite plausible that these results can be cobbled together using the powerful machines developed by Mazzeo [Maz91], Mazzeo and Vertman [MV14], and Albin and Gell-Redman [AG16; AG23]. However, the arguments in Section 4 are almost elementary and there should be some value in that.

It should be possible, with suitable modifications, to extend the work in the present article to higher rank ramified Euclidean local systems; in particular: to flat Hermitian line bundles. In fact, Ammann–Große have on-going work in progress in this direction and some instances of this appear in Portmann, Sok, and Solovej's work on magnetic links [PSS18b; PSS18a; PSS20].

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Conventions. *Choose* a cut-off function $\chi \in C^\infty([0,\infty),[0,1])$ with $\chi|_{[0,1/4]} = 1$ and $\operatorname{supp}(\chi) \subset [0,1/2)$. The **bracket** $\langle - \rangle \colon \mathbf{R} \to [1,\infty)$ is defined by $\langle x \rangle \coloneqq (1+x^2)^{1/2}$.

2 The Gelfand–Robbin quotient, I: abstract theory

This section studies the closed extensions of the Dirac operator $D\colon H^1\Gamma(X\backslash Z, S\otimes \mathfrak{l}) \to L^2\Gamma(X\backslash Z, S\otimes \mathfrak{l})$, considered as an unbounded operator, following [MS98, Exercise 2.17; BF98, §3; SW08, Appendix B; BS18, Exercises 6.3.3 and 6.5.11]. Throughout, assume the following analytic condition on the branching locus Z.

Hypothesis 2.1. *There is an $r \in C^\infty(X\backslash Z,(0,\infty))$, uniformly comparable to the Riemannian distance to Z, such that following* ***borderline Hardy inequality*** *holds: for every $\phi \in H^1\Gamma(X\backslash Z, S\otimes\mathfrak{l})$, $r^{-1}\phi \in L^2\Gamma(X\backslash Z, S\otimes \mathfrak{l})$ and*

$$\|r^{-1}\phi\|_{L^2} \lesssim \|\phi\|_{H^1}.$$

Remark 2.2. Hypothesis 2.1 holds if Z is a codimension two submanifold; see Takahashi [Tak15, Lemma 2.6] or Lemma 3.3. Moreover, it holds in the situation considered by Haydys, Mazzeo, and Takahashi [HMT23] where $Z \subset X$ is a graph embedded in a 3–manifold. ♣

2.1 The minimal and maximal extension

Proposition 2.3. *The bounded operator* $D\colon H^1\Gamma(X\backslash Z, S\otimes\mathfrak{l}) \to L^2\Gamma(X\backslash Z, S\otimes\mathfrak{l})$ *is left semi-Fredholm; that is:* $\ker D$ *is finite-dimensional and* $\operatorname{im} D$ *is closed.*

The proof relies on the following consequences of the borderline Hardy inequality.

Lemma 2.4. *The following hold:*

(1) $H^1\Gamma(X\backslash Z, S\otimes\mathfrak{l}) = H^1_0\Gamma(X\backslash Z, S\otimes\mathfrak{l})$.

(2) *The inclusion* $H^1\Gamma(X\backslash Z, S\otimes\mathfrak{l}) \hookrightarrow L^2\Gamma(X\backslash Z, S\otimes\mathfrak{l})$ *is a compact operator.*

Proof. For $\varepsilon > 0$ set $\chi_\varepsilon \coloneqq \chi(r/\varepsilon)$. Let $\phi \in H^1\Gamma(X\backslash Z, S\otimes\mathfrak{l})$. Since $|r\mathrm{d}\chi_\varepsilon| \lesssim 1$,

$$\|\nabla(\chi_\varepsilon\phi)\|_{L^2} \leqslant \|(r\mathrm{d}\chi_\varepsilon)r^{-1}\phi\|_{L^2} + \|\chi_\varepsilon\nabla\phi\|_{L^2} \lesssim \left(\int_{\operatorname{supp}(\mathrm{d}\chi_\varepsilon)} |r^{-1}\phi|^2 + |\nabla\phi|^2\right)^{1/2}.$$

Therefore, by Hypothesis 2.1 and monotone convergence,

$$\lim_{\varepsilon\downarrow 0}\|\nabla(\chi_\varepsilon\phi)\|_{L^2} = 0.$$

This implies (1).

Let $(\phi_n) \in H^1\Gamma(X\backslash Z, S\otimes\mathfrak{l})^{\mathbf{N}}$ with $\|\phi_n\|_{H^1} = 1$ for every $n \in \mathbf{N}$. For every $\varepsilon > 0$, a subsequence of $((1-\chi_\varepsilon)\phi_n)$ converges in $L^2\Gamma(X\backslash Z, S\otimes\mathfrak{l})$. By the borderline Hardy inequality,

$$\|\chi_\varepsilon\phi_n\|_{L^2} \lesssim \varepsilon.$$

Therefore, (2) follows from a diagonal sequence argument. ■

The proof of Proposition 2.3 also uses the following observation.

Proposition 2.5 (cf. Bismut [Bis89, Theorem 1.10]).

(1) D *is formally self-adjoint; in fact: for every* $\phi, \psi \in H^1_{\mathrm{loc}}\Gamma(X, S\otimes\mathfrak{l})$

$$\langle D\phi, \psi\rangle - \langle\phi, D\psi\rangle = \operatorname{div}(v) \quad \textit{with} \quad v \coloneqq \sum_{i=1}^n \langle\gamma(e_i)\phi, \psi\rangle \cdot e_i.$$

Here $(e_1, \dots, e_n)$ *denotes a local orthonormal frame.*

(2) D *satisfies*

$$D^2 = \nabla^*\nabla + \tau\nabla + \gamma(F_\nabla)$$

with $\tau \in \Gamma(X, \operatorname{Hom}(T^*X\otimes S, S))$ *depending linearly on* Tor.

Proof. The following argument can be found in [Bis89, Proof of Theorem 1.10] and is repeated here only for the readers' convenience.

By direct computation,

$$\operatorname{div} v = \sum_{i=1}^n \mathscr{L}_{e_i}\langle v, e_i\rangle = \langle D\phi, \psi\rangle - \langle\phi, D\psi\rangle + \sum_{i=1}^n \langle\gamma(\nabla^T_{e_i}e_i)\phi, \psi\rangle$$

and

$$\langle \nabla^T_{e_i} e_i, - \rangle = \tfrac{1}{2} \operatorname{Tor}(e_i, e_i, -) = 0.$$

This proves (1).

By direct computation,

$$\begin{aligned} D^2 = \sum_{i,j=1}^{n} \gamma(e_i)\nabla_{e_i}\gamma(e_j)\nabla_{e_j} &= \sum_{i,j=1}^{n} \gamma(e_j)\gamma(e_j)\nabla_{e_i}\nabla_{e_j} + \gamma(e_i)\gamma(\nabla^T_{e_i}e_j)\nabla_{e_j} \\ &= \nabla^*\nabla + \gamma(F_\nabla) + \gamma(e_i)\gamma(\nabla^T_{e_i}e_j)\nabla_{e_j} \end{aligned}$$

and

$$\langle \nabla^T_{e_i} e_j, e_k \rangle = \tfrac{1}{2} \operatorname{Tor}(e_i, e_j, e_k).$$

This proves (2). ■

Proof of Proposition 2.3. By Lemma 2.4 (1) and Proposition 2.5,

$$\|\phi\|_{H^1} \lesssim \|D\phi\|_{L^2} + \|\phi\|_{L^2} \tag{2.6}$$

for every $\phi \in H^1\Gamma(X\backslash Z, S \otimes \mathfrak{l})$. Therefore and by Lemma 2.4 (2), D is left semi-Fredholm. ■

With the exception of a few edge cases–e.g.: if $Z = \varnothing$ or $Z \subset X$ is a finite subset of a surface [DW24, §3.4.2; HMT23, §4.1–4.5]—the operator $D\colon H^1\Gamma(X\backslash Z, S \otimes \mathfrak{l}) \to L^2\Gamma(X\backslash Z, S \otimes \mathfrak{l})$ is not Fredholm: its cokernel is ∞–dimensional. Therefore, it is useful to consider D as an unbounded operator and systematically study its closed extensions; cf. [BS18, Chapter 6].

Definition 2.7. The **minimal extension**

$$D_{\min}\colon \operatorname{dom}(D_{\min}) \to L^2\Gamma(X\backslash Z, S \otimes \mathfrak{l});$$

is the operator $D\colon H^1\Gamma(X\backslash Z, S \otimes \mathfrak{l}) \to L^2\Gamma(X\backslash Z, S \otimes \mathfrak{l})$ considered as unbounded operator on $L^2\Gamma(X\backslash Z, S \otimes \mathfrak{l})$. •

Proposition 2.8. *$D_{\min}$ is closed, densely defined, and symmetric.*

Proof. Evidently, $D_{\min}$ is densely defined. By Proposition 2.5 (1),

$$\langle D\phi, \psi \rangle = \langle \phi, D\psi \rangle$$

for every $\phi, \psi \in \Gamma_c(X\backslash Z, S \otimes \mathfrak{l})$. Therefore, by Lemma 2.4 (1) and since $\Gamma_c(X\backslash Z, S \otimes \mathfrak{l}) \subset H^1_0\Gamma(X\backslash Z, S \otimes \mathfrak{l})$ is dense, $D_{\min}$ is symmetric. By (2.6) the Sobolev norm $\|-\|_{H^1}$ and the graph norm $\|-\|_D \coloneqq (\|-\|^2_{L^2} + \|D-\|^2_{L^2})^{1/2}$ are equivalent. Therefore and since $H^1\Gamma(X\backslash Z, S \otimes \mathfrak{l})$ is complete, $D_{\min}$ is closed. ■

Definition 2.9. The **maximal extension**

$$D_{\max}\colon \operatorname{dom}(D_{\max}) \to L^2(X\backslash Z, S \otimes \mathfrak{l})$$

is the adjoint of $D_{\min}$ in the sense of unbounded operators; that is:

$$\operatorname{dom}(D_{\max}) \coloneqq \left\{\phi \in L^2\Gamma(X\backslash Z, S \otimes \mathfrak{l}) : \langle \phi, D_{\min} - \rangle_{L^2} \colon \operatorname{dom}(D_{\min}) \to \mathbf{R} \text{ is } \|-\|_{L^2}\text{-bounded}\right\}$$

and for every $\phi \in \operatorname{dom}(D_{\max})$ and $\psi \in \operatorname{dom}(D_{\min})$

$$\langle D_{\max}\phi, \psi \rangle = \langle \phi, D_{\min}\psi \rangle.$$

$D_{\max}\phi$ exists by the Hahn–Banach Theorem and the Riesz Representation Theorem, and is unique because $\operatorname{dom}(D_{\min})$ is dense. •

Remark 2.10. It is convenient to consider $D \colon H^1_{\mathrm{loc}}\Gamma(X\backslash Z, S \otimes \mathfrak{l}) \to L^2_{\mathrm{loc}}\Gamma(X\backslash Z, S \otimes \mathfrak{l})$. From this perspective,

$$\operatorname{dom}(D_{\max}) = \{\phi \in H^1_{\mathrm{loc}}\Gamma(X\backslash Z, S \otimes \mathfrak{l}) : \phi, D\phi \in L^2\Gamma(X\backslash Z, S \otimes \mathfrak{l})\};$$

and it is excusable to drop the subscripts from $D_{\min}\phi$, $D_{\max}\phi$, etc. ♣

2.2 Closed extensions and residue conditions

The closed extensions of $D_{\min}$ can be systematically understood as follows.

Definition 2.11. The **Gelfand–Robbin quotient** is the Hilbert space

$$\check{\mathbf{H}} \coloneqq \frac{\operatorname{dom}(D_{\max})}{\operatorname{dom}(D_{\min})}.$$

Since $D_{\min}$ is closed, $\operatorname{dom}(D_{\min}) \subset \operatorname{dom}(D_{\max})$ is a $\|-\|_D$–closed subspace. Denote the canonical projection map by

$$[\cdot] \colon \operatorname{dom}(D_{\max}) \to \check{\mathbf{H}}.$$ •

Remark 2.12. $\check{\mathbf{H}}$ is localised on Z in the following sense: $[\phi] = [\chi(r/\varepsilon)\phi]$ for every $\varepsilon > 0$ and $\phi \in \operatorname{dom}(D_{\max})$. ♣

Definition 2.13. A **residue condition** is a closed subspace $B \subset \check{\mathbf{H}}$. •

Proposition 2.14 (closed extension=residue condition; cf. [BF98, Lemma 3.3(a)]). *If $B \subset \check{\mathbf{H}}$ is a residue condition, then*

$$D_B \coloneqq D_{\max}|_{\operatorname{dom}(D_B)} \quad \textit{with} \quad \operatorname{dom}(D_B) \coloneqq [\cdot]^{-1}(B)$$

is a closed extension of $D_{\min}$. Moreover, every closed extension of $D_{\min}$ is of this form.

Proof. Let $B \subset \check{\mathbf{H}}$ be a residue condition. The canonical projection $[\cdot] \colon \operatorname{dom}(D_{\max}) \to \check{\mathbf{H}}$ is bounded. Therefore, $\operatorname{dom}(D_B) \coloneqq [\cdot]^{-1}(B) \subset \operatorname{dom}(D_{\max})$ is a $\|-\|_D$–closed subspace; hence: D_B is closed.

Let $\bar{D}$ be a closed extension of $D_{\min}$. Since $\operatorname{dom}(\bar{D}) \subset \operatorname{dom}(D_{\max})$ is a $\|-\|_D$–closed subspace, $B \coloneqq [\operatorname{dom}(\bar{D})] = \frac{\operatorname{dom}(\bar{D})}{\operatorname{dom}(D_{\min})} \subset \check{\mathbf{H}}$ is a closed subspace. Since $\operatorname{dom}(D_{\min}) \subset \operatorname{dom}(\bar{D})$, $\operatorname{dom}(\bar{D}) = [\cdot]^{-1}(B)$; hence: $\bar{D} = D_B$. ■

2.3 The Green's form and adjoint extensions

The Gelfand–Robbin quotient carries a symplectic structure related to the construction of adjoint extensions.

Definition 2.15. The **Green's form** $G \in \operatorname{Hom}(\Lambda^2\check{\mathbf{H}}, \mathbf{R})$ is defined by

$$G([\phi] \wedge [\psi]) \coloneqq \langle D\phi, \psi\rangle_{L^2} - \langle \phi, D\psi\rangle_{L^2}. \qquad \bullet$$

Proposition 2.16 (cf. [BF98, Lemma 3.1, Proposition 3.2; SW08, Remark B.1(ii)]). *G is symplectic; that is: it induces a Hilbert space isomorphism*

$$\begin{aligned} J\colon\ \check{\mathbf{H}} &\to \hat{\mathbf{H}} \coloneqq \mathscr{L}(\check{\mathbf{H}}, \mathbf{R}) \\ [\phi] &\mapsto G([\phi] \wedge -). \end{aligned}$$

Moreover: if $\sharp\colon\ \hat{\mathbf{H}} \to \check{\mathbf{H}}$ denotes the isomorphism induced by the inner product, then $\sharp \circ J$ is an isometric complex structure.

Proof. The canonical projection induces an isometry $[-]\colon\ \operatorname{dom}(D_{\min})^{\perp_D} \cong \check{\mathbf{H}}$. Here $\perp_D$ indicates the orthogonal complement with respect to the graph inner product $\langle -,-\rangle_D \coloneqq \langle -,-\rangle_{L^2} + \langle D-, D-\rangle_{L^2}$. By direct inspection,

$$\begin{aligned} \operatorname{dom}(D_{\min})^{\perp_D} &= \{\phi \in \operatorname{dom}(D_{\max}) : \langle \phi, \psi\rangle_{L^2} + \langle D\phi, D\psi\rangle_{L^2} = 0 \text{ for every } \psi \in \operatorname{dom}(D_{\min})\} \\ &= \{\phi \in \operatorname{dom}(D_{\max}) : D\phi \in \operatorname{dom}(D_{\max}) \text{ and } D^2\phi = -\phi\}. \end{aligned}$$

Therefore, D induces an isometric complex structure $D\colon\ \operatorname{dom}(D_{\min})^{\perp_D} \to \operatorname{dom}(D_{\min})^{\perp_D}$.

The diagram

$$\begin{tikzcd} \operatorname{dom}(D_{\min})^{\perp_D} \arrow[r, "D"] \arrow[d, "{[-]}"'] & \operatorname{dom}(D_{\min})^{\perp_D} \arrow[d, "{[-]}"] \\ \check{\mathbf{H}} \arrow[r, "\sharp\circ J"'] & \check{\mathbf{H}} \end{tikzcd}$$

commutes because for every $\phi, \psi \in \operatorname{dom}(D_{\min})^{\perp_D}$

$$G([\phi] \wedge [\psi]) = \langle D\phi, \psi\rangle_{L^2} - \langle \phi, D\psi\rangle_{L^2} = \langle D\phi, \psi\rangle_{L^2} + \langle D^2\phi, D\psi\rangle_{L^2} = \langle D\phi, \psi\rangle_D.$$

This proves the assertion. ■

Proposition 2.17 (cf. [BF98, Lemma 3.3(b)]). *Let $B \subset \check{\mathbf{H}}$ be a residue condition. The adjoint D_B^* of D_B is the closed extension D_{B^G} associated with the symplectic complement*

$$B^G \coloneqq \{[\phi] \in \check{\mathbf{H}} : G([\phi], [\psi]) = 0 \text{ for every } [\psi] \in B\}.$$

In particular, D_B is self-adjoint if and only if B is Lagrangian.

Proof. A moment's thought shows that

$$\begin{aligned} \operatorname{dom}(D_B^*) &= \{\phi \in \operatorname{dom}(D_{\max}) : \langle D\phi, \psi\rangle_{L^2} = \langle \phi, D\psi\rangle_{L^2} \text{ for every } \psi \in \operatorname{dom}(D_B)\} \\ &= \{\phi \in \operatorname{dom}(D_{\max}) : G([\phi] \wedge [\psi]) = 0 \text{ for every } [\psi] \in B\}. \end{aligned}$$

This proves the assertion. ■

Example 2.18. The **Calderón subspace**

$$\Lambda \coloneqq [\ker D_{\max}] \subset \check{\mathbf{H}}$$

is a Lagrangian residue condition. Indeed, $\Lambda \subset \Lambda^G$ because $G([\phi] \wedge [\psi]) = \langle D\phi, \psi \rangle_{L^2} - \langle \phi, D\psi \rangle_{L^2} = 0$ for every $\phi, \psi \in \ker D_{\max}$. Moreover, if $[\phi] \in \Lambda^G$, then $D\phi \perp_{L^2} \ker D_{\max} = (\operatorname{im} D_{\min})^{\perp_{L^2}}$; therefore, there is a $\psi \in \operatorname{dom}(D_{\min})$ with $D\psi = D\phi$; hence: $[\phi] = [\phi - \psi] \in \Lambda$. ♠

Example 2.19. As a consequence of Proposition 2.16, the orthogonal complement of the Calderón subspace

$$\Lambda^\perp = \sharp \circ J(\Lambda) \subset \check{\mathbf{H}}$$

is a Lagrangian residue condition. ♠

Example 2.20. Suppose that S carries a parallel orthogonal complex structure i which commutes with γ; that is: (S, γ, ∇) is a **complex Dirac bundle with skew torsion**. Evidently, D is complex linear and $\check{\mathbf{H}}$ inherits i as an isometric complex structure i. This induces an orthogonal decomposition

$$\check{\mathbf{H}} = B_+ \oplus B_- \quad \text{with} \quad B_\pm \coloneqq \left\{ [\phi] \in \check{\mathbf{H}} : \sharp \circ J[\phi] = \pm i[\phi] \right\}.$$

Since i and $\sharp \circ J$ commute, $B_\pm \subset \check{\mathbf{H}}$ are complex subspaces and, therefore, $B_\pm$ are mutually adjoint:

$$B_\pm^G = B_\mp. \qquad ♠$$

Remark 2.21 (defect indices). If H is a *complex* Hilbert space and $A \colon \operatorname{dom}(A) \to H$ is a closed and symmetric unbounded *complex* linear operator, then its closed *complex linear* extensions traditionally are studied via von Neumann's theory of **defect subspaces** and **defect indices** [vNeu30, Kapitel VII; RS75, §X.1]. The defect subspaces of A are $\ker(A^* \mp i)$ and its defect indices are $n_\pm \coloneqq \dim \ker(A^* \mp i)$. The maximal domain orthogonally decomposes as

$$\operatorname{dom}(A^*) = \operatorname{dom}(A) \oplus \ker(A^* - i) \oplus \ker(A^* + i)$$

with respect to the graph inner product. Therefore, $\check{\mathbf{H}} \coloneqq \frac{\operatorname{dom}(A^*)}{\operatorname{dom}(A)} \cong \ker(A^* - i) \oplus \ker(A^* + i)$. In particular, closed self-adjoint complex linear extension of A correspond to closed complex linear Lagrangian subspaces $B \subset \ker(A^* - i) \oplus \ker(A^* + i)$. The latter exist if and only if $n_+ = n_-$. Of course, by Zorn's Lemma, $\check{\mathbf{H}}$ always has a (real) Lagrangian subspace. ♣

Proposition 2.22 (Spectral theory). *Let $B \subset \check{\mathbf{H}}$ be a residue condition. If $B \subset \check{\mathbf{H}}$ is a Lagrangian and* $\operatorname{dom}(D_B) \hookrightarrow L^2\Gamma(X\backslash Z, S \otimes \mathfrak{l})$ *is compact, then* $\operatorname{spec}(D_B)$ *consists only of point spectrum, is contained in* $\mathbf{R}$ *and discrete, and for every $\lambda \in \operatorname{spec}(D_B)$ the eigenspace* $\ker(D_B - \lambda \cdot \mathbf{1})$ *is finite-dimensional; moreover: $L^2\Gamma(X\backslash Z, S \otimes \mathfrak{l})$ decomposes as a (Hilbert space) direct sum*

$$L^2\Gamma(X\backslash Z, S \otimes \mathfrak{l}) = \bigoplus_{\lambda \in \operatorname{spec}(D_B)} \ker(D_B - \lambda \cdot \mathbf{1}).$$

Proof. By assumption, D_B is self-adjoint and has compact resolvent. The assertion, therefore, follows from the spectral theory of such operators; see, e.g., [BS18, Theorem 6.3.13]. ■

2.4 Fredholm extensions

The following characterises residue conditions $B \subset \check{\mathbf{H}}$ which correspond to Fredholm extensions D_B in terms of the relation between B and the Calderón subspace Λ.

Definition 2.23. Let $B \subset \check{\mathbf{H}}$ be a residue condition. Denote by

$$\delta_B \colon \Lambda \to \check{\mathbf{H}}/B \quad \text{and} \quad \delta^B \colon B \to \check{\mathbf{H}}/\Lambda$$

the compositions of the canonical inclusions and projections. •

Proposition 2.24 (cf. [SW08, Lemma B.3]). *Let $B \subset \check{\mathbf{H}}$ be a residue condition. The closed extension D_B is Fredholm if and only if δ_B is Fredholm if and only if δ^B is Fredholm; moreover:*

$$\operatorname{index} D_B = \operatorname{index} \delta_B = \operatorname{index} \delta^B.$$

The proof relies on the following observation.

Lemma 2.25. *For every residue condition $B \subset \check{\mathbf{H}}$, there are short exact sequences*

$$\ker D_{\min} \hookrightarrow \ker D_B \twoheadrightarrow \ker \delta_B \quad \textit{and} \quad \operatorname{coker} \delta_B \hookrightarrow \operatorname{coker} D_B \twoheadrightarrow \operatorname{coker} D_{\max}.$$

Proof. The Snake Lemma applied to

$$\begin{tikzcd}
\ker D_{\min} \arrow[r, equal] \arrow[d, hook] & \ker D_{\min} \arrow[d, hook] & \\
\ker D_B \arrow[r, hook] & \ker D_{\max} \arrow[r, "\delta_B \circ [-]"] & \check{\mathbf{H}}/B
\end{tikzcd}$$

yields an exact sequence

$$\frac{\ker D_B}{\ker D_{\min}} \hookrightarrow \frac{\ker D_{\max}}{\ker D_{\min}} \cong \Lambda \xrightarrow{\delta_B} \check{\mathbf{H}}/B.$$

This induces the first short exact sequence.

The Snake Lemma applied to

$$\begin{tikzcd}
\operatorname{dom}(D_B) \arrow[r, hook] \arrow[d, "D_B"] & \operatorname{dom}(D_{\max}) \arrow[r, two heads] \arrow[d, "D_{\max}"] & \check{\mathbf{H}}/B \\
L^2\Gamma(X \backslash Z, S \otimes \mathfrak{l}) \arrow[r, equal] & L^2\Gamma(X \backslash Z, S \otimes \mathfrak{l}) &
\end{tikzcd}$$

yields an exact sequence

$$\ker D_B \hookrightarrow \ker D_{\max} \xrightarrow{\delta_B \circ [-]} \check{\mathbf{H}}/B \to \operatorname{coker} D_B \twoheadrightarrow \operatorname{coker} D_{\max}.$$

Since $\operatorname{coker} \delta_B \circ [-] = \operatorname{coker} \delta_B$, this induces the second short exact sequence. ■

Proof of Proposition 2.24. A moment's thought shows that

$$\ker \delta_B = \Lambda \cap B = \ker \delta^B \quad \text{and} \quad \operatorname{coker} \delta_B \cong \frac{\check{\mathbf{H}}}{\Lambda + B} \cong \operatorname{coker} \delta^B.$$

By Proposition 2.3, $\ker D_{\min} \cong \operatorname{coker} D_{\max}$ is finite-dimensional. Therefore, the assertion is an immediate consequence of Lemma 2.25. ■

Example 2.26. Every complement B of the Calderón subspace Λ, in particular: $B = \Lambda^{\perp}$, produces a Fredholm extension of index 0 because $\delta^B \colon B \to \check{\mathbf{H}}/\Lambda \cong B$ is an isomorphism. ♠

Example 2.27. The residue conditions $B_{\pm}$ defined in Example 2.20 satisfy

$$\ker D_{B_{\pm}} = \ker D_{\min}$$

and, therefore, produce Fredholm extension of index 0; indeed: if $\phi \in \ker D_{B_{\pm}}$, then

$$0 = 2\langle D\phi, i\phi\rangle_{L^2} = \langle D\phi, i\phi\rangle_{L^2} - \langle \phi, Di\phi\rangle_{L^2} = G([\phi], i[\phi]) = \langle \sharp \circ J[\phi], i[\phi]\rangle_{L^2} = \pm\|[\phi]\|_{L^2}^2. \quad ♠$$

The following are occasionally useful to compute or relate indices.

Proposition 2.28 (Nested Fredholm residue conditions). *Let $B_1 \subset B_2 \subset \check{\mathbf{H}}$ be residue conditions. If $\delta_{B_1}, \delta_{B_2}$ are Fredholm, then*

$$\operatorname{index} D_{B_2} = \operatorname{index} D_{B_1} + \dim B_2/B_1.$$

Corollary 2.29. *Let $B \subset \check{\mathbf{H}}$ be a residue condition. If $B \subset B^G$ and δ_B is Fredholm then* $\operatorname{index} D_B = -\frac{1}{2}\dim B^G/B$*; in particular: if B is Lagrangian, then* $\operatorname{index} D_B = 0$. ■

Proposition 2.28 is an immediate consequence of the following.

Lemma 2.30 (Nested residue conditions). *Let $B_1 \subset B_2 \subset \check{\mathbf{H}}$ be residue conditions. There is an exact sequence*

$$\ker \delta_{B_1} \hookrightarrow \ker \delta_{B_2} \to B_2/B_1 \to \operatorname{coker} \delta_{B_1} \twoheadrightarrow \operatorname{coker} \delta_{B_2}.$$

Proof. The exact sequence follows from the Snake Lemma applied to

$$\begin{tikzcd}
& \Lambda \arrow[r, equal] \arrow[d, "\delta_{B_1}"] & \Lambda \arrow[d, "\delta_{B_2}"] \\
B_2/B_1 \arrow[r, hook] & \check{\mathbf{H}}/B_1 \arrow[r, two heads] & \check{\mathbf{H}}/B_2.
\end{tikzcd}$$

This implies the assertion by the above proposition. ■

Proposition 2.31 (Deformation of residue conditions). *Let B be a Hilbert space. Let $\iota_{-} \colon [0,1] \to \mathscr{L}(B, \check{\mathbf{H}})$ be a continuous path of embeddings. If D_{B_t} with $B_t \coloneqq \iota_t(B)$ is Fredholm for every $t \in [0,1]$, then*

$$\operatorname{index} D_{B_0} = \operatorname{index} D_{B_1}.$$

Proof. By assumption, $t \mapsto \delta^{B_t} \circ \iota_t$ is a continuous path of Fredholm operators and $\iota_t \colon B \to B_t$ is an isomorphism for every $t \in [0,1]$; therefore: $t \mapsto \operatorname{index} D_{B_t} = \operatorname{index} \delta^{B_t} = \operatorname{index} \delta^{B_t} \circ \iota_t$ is constant. ■

2.5 Chirality operators

In the presence of a chirality operator the theory discussed above refines as follows.

Definition 2.32. A **chirality operator** on (S, γ, ∇) is a self-adjoint parallel isometry $\varepsilon \in \Gamma(X, \mathrm{O}(S))$ such that

$$\gamma\varepsilon + \varepsilon\gamma = 0. \qquad \bullet$$

Example 2.33. Assume that X is oriented. If $\dim X = 0 \bmod 4$, then $\varepsilon \coloneqq \gamma(\mathrm{vol}_g)$ is a chirality operator. If $\dim X = 0 \bmod 2$ and (S, γ, ∇) is a complex Dirac bundle as in Example 2.20, then $\varepsilon \coloneqq i^{\lfloor (n+1)/2 \rfloor}\gamma(\mathrm{vol}_g)$ is a chirality operator. ♠

Proposition 2.34 (Chirality operators induce a $\{\pm 1\}$–grading). *If ε is a chirality operator for (S, γ, ∇), then the following hold:*

(1) *The subbundles $S^\pm \coloneqq \ker(\mathbf{1} \pm \varepsilon) \subset S$ are parallel, S orthogonally decomposes as*

$$S = S^+ \oplus S^-,$$

and $\gamma \in \Gamma(X, \mathrm{Hom}(TX, \mathrm{Hom}(S^+, S^-) \oplus \mathrm{Hom}(S^-, S^+)))$.

(2) *The minimal and maximal extensions decompose as*

$$D_{\min} = \begin{pmatrix} 0 & D_{\min}^- \\ D_{\min}^+ & 0 \end{pmatrix} \quad \textit{and} \quad D_{\max} = \begin{pmatrix} 0 & D_{\max}^- \\ D_{\max}^+ & 0 \end{pmatrix}$$

with

$$\begin{aligned} D_{\min}^\pm \colon \ \mathrm{dom}(D_{\min}^\pm) &\coloneqq H^1\Gamma(X\backslash Z, S^\pm \otimes \mathfrak{l}) \to L^2\Gamma(X\backslash Z, S^\mp \otimes \mathfrak{l}) \quad \textit{and} \\ D_{\max}^\pm \colon \ \mathrm{dom}(D_{\max}^\pm) &\coloneqq \mathrm{dom}(D_{\max}) \cap L^2\Gamma(X\backslash Z, S^\pm \otimes \mathfrak{l}) \to L^2\Gamma(X\backslash Z, S^\mp \otimes \mathfrak{l}). \end{aligned}$$

(3) $\check{\mathbf{H}}$ *orthogonally decomposes as*

$$\check{\mathbf{H}} = \check{\mathbf{H}}^+ \oplus \check{\mathbf{H}}^- \quad \textit{with} \quad \check{\mathbf{H}}^\pm \coloneqq \frac{\mathrm{dom}(D_{\max}^\pm)}{\mathrm{dom}(D_{\min}^\pm)};$$

moreover, $\check{\mathbf{H}}^\pm \subset \check{\mathbf{H}}$ are Lagrangian. In particular, every residue condition $B \subset \check{\mathbf{H}}$ decomposes as $B = B^+ \oplus B^-$.

(4) *If $B^+ \subset \check{\mathbf{H}}^+$ is a closed subspace, a **positive chirality residue condition**, then there is a unique closed subspace $B^- \subset \check{\mathbf{H}}^-$ such that $B \coloneqq B^+ \oplus B^- \subset \check{\mathbf{H}}$ is a Lagrangian residue condition.*

(5) *Let $B^+ \subset \check{\mathbf{H}}^+$ be a **positive chirality residue condition** and B^- as above. The operator δ^B is Fredholm if and only if $\delta^{B_+} \colon \ B^+ \to \check{\mathbf{H}}^+/\Lambda^+$ is Fredholm.*

Proof. (1), (2), and (3) are an immediate consequence of Definition 2.32.

A moment's thought shows that (4) holds with $B^- \coloneqq (B^+)^G \cap \check{\mathbf{H}}^-$.

Evidently, $\delta^B = \delta^{B_+} \oplus \delta^{B_-}$ is Fredholm if and only if δ^{B_+} and δ^{B_-} are Fredholm. The Green's form G induces isomorphisms

$$\Lambda^- \cong (\check{\mathbf{H}}^+/\Lambda^+)^* \quad \text{and} \quad B^+ \cong (\check{\mathbf{H}}^-/B^-)^*.$$

This identifies the dual of δ^{B^+} with $\delta_{B_-} \colon \Lambda^- \to \check{\mathbf{H}}^-/B^-$. By the closed image theorem, if δ^{B^+} is Fredholm, then δ_{B_-} is Fredholm. As in the proof of Proposition 2.24, δ_{B_-} is Fredholm if and only if δ^{B_-} is Fredholm. This proves (5). ■

3 The Gelfand–Robbin quotient, II: geometric realisation

The usefulness of the theory laid out in Section 2 hinges upon being able to understand $\check{\mathbf{H}}$, e.g., to specify interesting residue conditions. Since $\check{\mathbf{H}}$ localises on Z as explained in Remark 2.12, it is plausible that it admits a more geometric description. The purpose of this section is to develop such a description, assuming the following geometric condition on the branching locus Z throughout.

Hypothesis 3.1. *$Z \subset X$ is a closed cooriented submanifold of codimension two.*

Remark 3.2. The assumption that Z is cooriented simplifies the upcoming discussion, but is not essential. Appendix A explains how to remove it. ♣

Lemma 3.3 (borderline Hardy inequality; Takahashi [Tak15, Lemma 2.6]). *Hypothesis 3.1 implies Hypothesis 2.1.*

Proof. Let $r > 0$. Denote by $\mathfrak{l}$ the non-trivial Euclidean line bundle over $rS^1 \coloneqq \{z \in \mathbf{C} : |z| = r\}$. A moment's thought and a scaling consideration show that

$$\int_{rS^1} |r^{-1}s|^2 \lesssim \int_{rS^1} |\nabla s|^2$$

for every $s \in \Gamma(rS^1, \mathfrak{l})$. This immediately implies the assertion. ■

3.1 The blow-up of X along Z

It is convenient to blow-up X along Z; that is: to replace $Z \subset X$ with the following $\mathrm{U}(1)$–principal bundle.

Definition 3.4. Since Z is cooriented, its normal bundle NZ is a Hermitian line bundle over Z. Its **frame bundle** is

$$\pi \colon F \coloneqq \{v \in NZ : |v| = 1\} \to Z$$

together with $F \circlearrowleft \mathrm{U}(1)$ defined by $v \cdot e^{i\alpha} \coloneqq e^{i\alpha}v$. Denote the Levi-Civita connection on F by $i\theta \in \Omega^1(F, i\mathbf{R})$. •

Remark 3.5. The **tautological section** $\partial_r \in \Gamma(F, \pi^*NZ)$ and $\partial_\alpha \coloneqq i\partial_r$ trivialise π^*NZ. ♣

In order to replace $Z \subset X$ with F a choice is required.

Definition 3.6. Set $U \coloneqq [0,1) \cdot F \subset NZ$. A **tubular neighbourhood** $\jmath \colon U \hookrightarrow X$ of $Z \subset X$ is an embedding such that $\jmath \circ 0 = \mathrm{id}_Z$ and the composition

$$NZ \hookrightarrow 0^* TNZ \xrightarrow{T\jmath} TX|_Z \twoheadrightarrow NZ$$

is the identity. Here $0 \colon Z \to NZ$ denotes the zero section. •

Choose a tubular neighbourhood $\jmath \colon U \hookrightarrow X$.

Definition 3.7. Set $\hat{U} \coloneqq [0,1) \times F$. The **blow-up** of X along Z is the manifold with boundary

$$\hat{X} \coloneqq \hat{U} \cup_\jmath (X \backslash Z)$$

obtained by gluing $\hat{U}$ and $X\backslash Z$ along $\jmath$. The **blow-down map** $\beta \colon \hat{X} \to X$ is defined by $\beta(r,v) \coloneqq \jmath(rv)$ for $(r,v) \in \hat{U}$ and $\beta(x) \coloneqq x$ for $x \in X\backslash Z$. •

Henceforth, identify $U \subset NZ$ and $\jmath(U) \subset X$; moreover, identify $\partial\hat{X} = F$.

Definition 3.8. Set

$$\hat{S} \coloneqq \beta^* S \quad \text{and} \quad \underline{S} \coloneqq \hat{S}|_F = \pi^*(S|_Z).$$

Endow $S|_Z$ with the complex structure $I \coloneqq \gamma(\mathrm{vol}_{NZ})$ and $\underline{S}$ with the quaternionic structure

$$I \coloneqq \gamma(\mathrm{vol}_{NZ}), \quad J \coloneqq \gamma(\partial_r), \quad \text{and} \quad K = IJ \coloneqq \gamma(\partial_\alpha) \in \Gamma(F, \mathrm{End}(\underline{S})).$$

Since $X\backslash Z \hookrightarrow \hat{X}$ is a homotopy-equivalence, $\mathfrak{l}$ extends uniquely to a Euclidean line bundle

$$\hat{\mathfrak{l}} \to \hat{X}.$$

Set

$$\underline{\mathfrak{l}} \coloneqq \hat{\mathfrak{l}}|_F.$$ •

3.2 The model operator

The purpose of this subsection is to construct a model $\mathring{D}$ for D near Z. This construction relies on the following.

Definition 3.9 (Restriction of Dirac bundles). Denote the second fundamental form of Z with respect to ∇^T by $\mathrm{II} \in \Gamma(Z, \mathrm{Hom}(TZ, \mathrm{Hom}(TZ, NZ)))$. The **restriction** of $(S, \gamma, \nabla, \mathrm{Tor})$ to Z is the quadruple $(S|_Z, \gamma|_{TZ}, \nabla|_Z + \frac{1}{2}\gamma(\mathrm{II}), \mathrm{Tor}\,|_Z)$ with

$$\gamma(\mathrm{II})(v) \coloneqq \sum_{i=1}^{n-2} \gamma(\mathrm{II}(v)e_i)\gamma(e_i).$$

Here $(e_1, \dots, e_{n-2})$ denotes a local orthonormal frame of TZ. •

Proposition 3.10. *$(S|_Z, \gamma|_{TZ}, \nabla|_Z + \frac{1}{2}\gamma(\mathrm{II}), \mathrm{Tor}\,|_Z)$ is a Dirac bundle with skew torsion over $(Z, g|_Z)$.*

Proof. Evidently, $(S|_Z, \gamma|_{TZ})$ forms a Clifford module bundle over $(Z, g|_Z)$. Denote by $\nabla^{T,\parallel}$ the orthogonal affine connection on $(Z, g|_Z)$ induces by ∇^T.

Since $(S, \gamma, \nabla, \mathrm{Tor})$ is a Dirac bundle with skew torsion over (X, g), for every $v, w \in \mathrm{Vect}(Z)$

$$[\nabla_v, \gamma(w)] = \gamma(\nabla_v^{T,\|} w) + \gamma(\mathrm{II}(v)w);$$

moreover, by direct computation,

$$\begin{aligned}[\gamma(\mathrm{II})(v), \gamma(w)] = \sum_{i=1}^{n-2} [\gamma(\mathrm{II}(v)e_i)\gamma(e_i), \gamma(w)] &= \sum_{i=1}^{n-2} \gamma(\mathrm{II}(v)e_i)(\gamma(e_i)\gamma(w) + \gamma(w)\gamma(e_i)) \\ &= -2\sum_{i=1}^{n-2} \gamma(\mathrm{II}(v)e_i)\langle e_i, w\rangle = -2\gamma(\mathrm{II}(v)w).\end{aligned}$$

A moment's thought shows that if $\nabla^{\|}$ the Levi-Civita connection of $(Z, g|_Z)$, then

$$\langle \nabla_u^{T,\|} v, w\rangle = \langle \nabla_u^{\|} v, w\rangle + (\tfrac{1}{2}\,\mathrm{Tor}\,|_Z)(u, v, w).$$

This proves the assertion. ■

Proposition 3.11. *Denote by* $g^{\|} \coloneqq g|_Z$ *and* $g^{\perp}$ *the Euclidean metrics on* TZ *and* NZ *induced by* g*. Denote by* $\Pi\colon\ U \longrightarrow Z$ *the projection map and identify* $TU = \Pi^*(TZ \oplus NZ)$ *using the Levi-Civita connection. Consider* $U \subset X$ *equipped with the Riemannian metric*

$$\mathring{g} \coloneqq \Pi^*(g^{\|} \oplus g^{\perp}).$$

The quadruple $(\mathring{S}, \mathring{\gamma}, \mathring{\nabla}, \mathring{\mathrm{Tor}})$ *consisting of*

$$\begin{gathered}\mathring{S} \coloneqq \Pi^*(S|_Z), \quad \mathring{\gamma} \coloneqq \Pi^*(\gamma|_Z), \quad \mathring{\nabla} \coloneqq \Pi^*(\nabla|_Z + \tfrac{1}{2}\gamma(\mathrm{II})), \\ \text{and} \quad \mathring{\mathrm{Tor}} \coloneqq \Pi^*(\mathrm{Tor}_Z^{3,0} + \mathrm{Tor}_Z^{1,2}) + r\mathrm{pr}_F^*(\mathrm{d}\theta \wedge \theta)\end{gathered}$$

is a Dirac bundle with skew torsion over $(U, \mathring{g})$*. Here* $\mathrm{Tor}_Z^{p,q}$ *denotes the* (p, q) *component with respect to* $\Lambda^\bullet(T^*Z \oplus N^*Z) = \Lambda^\bullet T^*Z \otimes \Lambda^\bullet N^*Z$ *of the restriction of* Tor *to* Z*.*

Proof. Denote by $\nabla^{T,\|}$ and $\nabla^{T,\perp}$ the orthogonal covariant derivatives on TZ and NZ induced by ∇^T respectively. If $v \in \mathrm{Vect}(Z)$ and $w \in \Gamma(Z, NZ)$, then

$$[\nabla_v, \gamma(w)] = \gamma(\nabla_v^{T,\perp} w) - \gamma(\mathrm{II}(v)^* w))$$

and, moreover,

$$\begin{aligned}[\gamma(\mathrm{II})(v), \gamma(w)] = \sum_{i=1}^{n-2} [\gamma(\mathrm{II}(v)e_i)\gamma(e_i), \gamma(w)] &= -\sum_{i=1}^{n-2} (\gamma(\mathrm{II}(v)e_i)\gamma(w) + \gamma(w)\gamma(\mathrm{II}(v)e_i))\gamma(e_i) \\ &= 2\sum_{i=1}^{n-2} \langle \mathrm{II}(v)e_i, w\rangle\gamma(e_i) = 2\gamma(\mathrm{II}^*(v)w)\end{aligned}$$

This together with the analogous computation in the proof of Proposition 3.10 proves that $\mathring{\gamma}$ is parallel with respect to $\mathring{\nabla}$ and $\Pi^*(\nabla^{T,\|} \oplus \nabla^{T,\perp})$. Therefore, it remains to identify the torsion of $\mathring{\nabla}^T \coloneqq \Pi^*(\nabla^{T,\|} \oplus \nabla^{T,\perp})$.

Denote by $\tilde{\cdot}\colon \operatorname{Vect}(Z) \to \operatorname{Vect}(U\backslash Z)$ and $\tilde{\cdot}\colon \Gamma(Z, NZ) \to \operatorname{Vect}(U\backslash Z)$ the lifting maps. For $u, v \in \operatorname{Vect}(Z)$ and $n, m \in \Gamma(Z, NZ)$, by direct computation,

$$\begin{aligned}\mathring{\nabla}^T_{\tilde{u}}\tilde{v} - \mathring{\nabla}^T_{\tilde{v}}\tilde{u} - [\tilde{u}, \tilde{v}] &= \operatorname{Tor}_Z^{3,0}(u, v, -)^\sharp + \widetilde{[u, v]} - [\tilde{u}, \tilde{v}] \\ &= \operatorname{Tor}_Z^{3,0}(u, v, -)^\sharp + (\operatorname{pr}_F^* \mathrm{d}\theta)(\tilde{u}, \tilde{v}) \otimes \partial_\alpha;\end{aligned}$$

moreover,

$$\mathring{\nabla}^T_{\tilde{n}}\tilde{m} - \mathring{\nabla}^T_{\tilde{m}}\tilde{n} - [\tilde{n}, \tilde{m}] = 0 \quad \text{and} \quad \mathring{\nabla}^T_{\tilde{v}}\tilde{n} - \mathring{\nabla}^T_{\tilde{n}}\tilde{v} - [\tilde{v}, \tilde{n}] = \operatorname{Tor}_Z^{1,2}(v, n, -)^\sharp.$$

This proves the assertion. ■

Definition 3.12. Denote by $\mathring{\mathfrak{I}}$ the pullback of $\underline{\mathfrak{I}}$ along the projection $U\backslash Z \cong F \times (0, 1) \to F$. The **model Dirac operator**

$$\mathring{D}\colon H^1_{\mathrm{loc}}\Gamma(U\backslash Z, \mathring{S} \otimes \mathring{\mathfrak{I}}) \to L^2_{\mathrm{loc}}\Gamma(U\backslash Z, \mathring{S} \otimes \mathring{\mathfrak{I}})$$

is the Dirac operator associated with $(\mathring{S}, \mathring{\nabla}, \mathring{\gamma}, \mathring{\operatorname{Tor}})$ twisted by $\mathring{\mathfrak{I}}$. •

Remark 3.13. More explicity, the model Dirac operator $\mathring{D}$ is of the form

$$\mathring{D} = J(\partial_r - r^{-1}I\mathring{\nabla}_{\partial_\alpha}) + D_Z \quad \text{with} \quad D_Z \coloneqq \sum_{i=1}^{n-2} \mathring{\gamma}(\tilde{e}_i)\mathring{\nabla}_{\tilde{e}_i}$$

with $(\tilde{e}_i, \dots, \tilde{e}_{n-2})$ denoting the horizontal lift of a local $g^{\|}$–orthonormal frame. ♣

Choose an isometry $\mathring{S} \cong S|_U$ which agrees with $\mathrm{id}_{S|_Z}$ over Z, and an isometry $\mathring{\mathfrak{I}} \cong \mathfrak{I}|_{U\backslash Z}$; moreover, *henceforth*, regard these as identifications.

Proposition 3.14. *The* ***error term***

$$\operatorname{Err} \coloneqq D - \mathring{D}\colon H^1_{\mathrm{loc}}\Gamma(U\backslash Z, \mathring{S} \otimes \mathring{\mathfrak{I}}) \to L^2_{\mathrm{loc}}\Gamma(U\backslash Z, \mathring{S} \otimes \mathring{\mathfrak{I}})$$

is of the form

$$\operatorname{Err} = a\mathring{\nabla} + b - \tfrac{1}{2}\Pi^*(\gamma(H_Z)) + \tfrac{1}{2}\Pi^*(\gamma(\operatorname{Tor}_Z^{2,1}))$$

with $a \in \Gamma(U, \operatorname{Hom}(T^*U \otimes \mathring{S}, \mathring{S}))$, $b \in \Gamma(U, \operatorname{End}(\mathring{S}))$, and H_Z denoting the mean curvature of Z. Moreover, a and b vanish along Z.

Proof. If $(e_1, \dots, e_{n-2})$ is a local orthonormal frame of TZ, then

$$\begin{aligned}-\frac{1}{2}\sum_{i=1}^{n-2} \gamma(e_i)\gamma(\mathrm{II})(e_i) &= \frac{1}{2}\sum_{i,j=1}^{n-2} \gamma(e_i)\gamma(e_j)\gamma(\mathrm{II}(e_i)e_j) \\ &= -\frac{1}{2}\gamma(H_Z) + \frac{1}{4}\sum_{i,j=1}^{n-2} \gamma(e_i)\gamma(e_j)\big(\gamma(\mathrm{II}(e_i)e_j) - \gamma(\mathrm{II}(e_j)e_i)\big) \\ &= -\frac{1}{2}\gamma(H_Z) + \frac{1}{2}\gamma(\operatorname{Tor}_Z^{2,1}).\end{aligned}$$

Therefore, the assertion follows from the fact that

$$\mathring{g} - g, \quad \mathring{\nabla} - \nabla - \tfrac{1}{2}\Pi^*(\gamma(\mathrm{II})), \quad \text{and} \quad \mathring{\gamma} - \gamma$$

vanish along Z. ■

3.3 The model Gelfand–Robbin quotient

By (the proof of) Proposition 2.8, the model minimal extension

$$\mathring{D}_{\min} \coloneqq \mathring{D}\colon\ \mathrm{dom}(\mathring{D}_{\min}) \coloneqq H^1_0\Gamma(U\backslash Z, \mathring{S} \otimes \mathring{\mathfrak{l}}) \to L^2\Gamma(U\backslash Z, \mathring{S} \otimes \mathring{\mathfrak{l}})$$

is closed, densely defined, and symmetric. A moment's thought shows that the domain of the model maximal extension

$$\mathring{D}_{\max} \coloneqq \mathring{D}^*_{\min}$$

is

$$\mathrm{dom}(\mathring{D}_{\max}) \coloneqq \{\phi \in H^1_{\mathrm{loc}}\Gamma(U\backslash Z, \mathring{S} \otimes \mathring{\mathfrak{l}}) : \phi, \mathring{D}\phi \in L^2\Gamma(U\backslash Z, \mathring{S} \otimes \mathring{\mathfrak{l}})\}.$$

The construction from Section 2.2 and Section 2.3 yields the following.

Definition 3.15. The **model Gelfand–Robbin quotient** is the Hilbert space

$$\mathring{\check{\mathbf{H}}} \coloneqq \frac{\mathrm{dom}(\mathring{D}_{\max})}{\mathrm{dom}(\mathring{D}_{\min})}$$

equipped with the **model Green's form** $\mathring{G} \in \mathrm{Hom}(\Lambda^2\mathring{\check{\mathbf{H}}}, \mathbf{R})$ defined by

$$\mathring{G}([\phi] \wedge [\psi]) \coloneqq \langle \mathring{D}\phi, \psi\rangle_{L^2} - \langle \phi, \mathring{D}\psi\rangle_{L^2}. \qquad \bullet$$

By (the proof of) Proposition 2.16, $(\mathring{\check{\mathbf{H}}}, \mathring{G})$ is a symplectic Hilbert space. In the sense of Remark 2.12, $\mathring{\check{\mathbf{H}}}$ has contributions from $\{0\} \times F \subset \hat{U}$ and $\{1\} \times F$. Only the former is relevant for the purposes of this section.

Proposition 3.16. *The subspace*

$$\mathring{\check{\mathbf{H}}}_0 \coloneqq \frac{\chi(r) \cdot \mathrm{dom}(\mathring{D}_{\max}) + \mathrm{dom}(\mathring{D}_{\min})}{\mathrm{dom}(\mathring{D}_{\min})} \subset \mathring{\check{\mathbf{H}}}$$

is closed and symplectic.

Proof. Define the operator $\pi \in \mathscr{L}(\mathring{\check{\mathbf{H}}})$ by $\pi([\phi]) \coloneqq [\chi(r)\cdot\phi]$. Since $\chi(r)(1-\chi(r))\cdot\mathrm{dom}(\mathring{D}_{\max}) \subset \mathrm{dom}(\mathring{D}_{\min})$, $\pi^2 = \pi$; that is: π is a projection. Hence, $\mathring{\check{\mathbf{H}}}_0 = \mathrm{im}\,\pi = \ker(\mathbf{1} - \pi)$ is closed.

Since $(1 - \chi(r) - \chi \circ (1 - r)) \cdot \mathrm{dom}(\mathring{D}_{\max}) \subset \mathrm{dom}(\mathring{D}_{\min})$, $(\mathbf{1} - \pi)[\phi] = [\chi \circ (1 - r) \cdot \phi]$. Therefore,

$$\mathring{G}([\phi] \wedge [\psi]) = \mathring{G}(\pi[\phi] \wedge \pi[\psi]) + \mathring{G}((\mathbf{1} - \pi)[\phi] \wedge (\mathbf{1} - \pi)[\psi]).$$

Hence, $\mathring{\check{\mathbf{H}}}_0$ is symplectic. ∎

Proposition 3.17. *There is a unique isomorphism of symplectic Hilbert spaces*

$$\text{cut-off}\colon\ (\check{\mathbf{H}}, G) \cong (\mathring{\check{\mathbf{H}}}_0, \mathring{G})$$

satisfying cut-off$([\phi]) = [\chi(r) \cdot \phi]$ *for every* $\phi \in \mathrm{dom}(D_{\max})$.

The proof requires the following preparation.

Lemma 3.18 (dom($D_{\max}$) vs. dom($\mathring{D}_{\max}$)). *The following hold:*

(1) *If* $\phi \in \operatorname{dom}(\mathring{D}_{\max})$, *then* $\chi(r)\cdot\phi \in \operatorname{dom}(D_{\max})$ *and* $\|\chi(r)\cdot\phi\|_D \lesssim \|\phi\|_{\mathring{D}}$.

(2) *If* $\phi \in \operatorname{dom}(D_{\max})$, *then* $\chi(r)\cdot\phi \in \operatorname{dom}(\mathring{D}_{\max})$ *and* $\|\chi(r)\cdot\phi\|_{\mathring{D}} \lesssim \|\phi\|_D$.

Proof. Let $\phi \in \operatorname{dom}(\mathring{D}_{\max})$. Let $\eta \in C_c^\infty(U\backslash Z, [0,1])$. By Proposition 2.5 (2),

$$\begin{aligned}\int_{U\backslash Z} \eta^2|\mathring{\nabla}(r\chi(r)\cdot\phi)|^2 = &\int_{U\backslash Z} \eta^2|\mathring{D}(r\chi(r)\cdot\phi)|^2\\ &-\int_{U\backslash Z} \eta^2\big(\langle\tau\mathring{\nabla}(r\chi(r)\cdot\phi), r\chi(r)\cdot\phi\rangle + \langle\mathring{\gamma}(F_{\mathring{\nabla}})r\chi(r)\cdot\phi, r\chi(r)\cdot\phi\rangle\big)\\ &+2\int_{U\backslash Z} \eta\langle\mathring{D}(r\chi(r)\cdot\phi), \mathring{\gamma}(\mathrm{d}\eta)r\chi(r)\cdot\phi\rangle\\ &-2\int_{U\backslash Z} \eta\langle\mathring{\nabla}(r\chi(r)\cdot\phi), \mathrm{d}\eta\otimes r\chi(r)\cdot\phi\rangle.\end{aligned}$$

Therefore,

$$\int_{U\backslash Z} \eta^2|\mathring{\nabla}(r\chi(r)\cdot\phi)|^2 \lesssim \int_{U\backslash Z} |\mathring{D}\phi|^2 + r^2(1+|\mathrm{d}\eta|^2)|\phi|^2.$$

Since $\eta_\varepsilon \coloneqq 1-\chi(r/\varepsilon)$ satisfies $r|\mathrm{d}\eta_\varepsilon| \lesssim 1$,

$$\int_{U\backslash Z} |\mathring{\nabla}(r\chi(r)\cdot\phi)|^2 = \lim_{\varepsilon\downarrow 0}\int_{U\backslash Z} \eta_\varepsilon^2|\mathring{\nabla}(r\phi)|^2 \lesssim \int_{U\backslash Z} |\mathring{D}\phi|^2 + |\phi|^2.$$

Therefore, $r\chi(r)\cdot\phi \in \operatorname{dom}(\mathring{D}_{\min})$ and

$$\|r\chi(r)\cdot\phi\|_{H^1} \lesssim \|\phi\|_{\mathring{D}}.$$

By Proposition 3.14 and the above,

$$\|\mathrm{Err}\,\chi(r)\cdot\phi\|_{L^2} \lesssim \|\mathring{\nabla}(r\phi)\|_{L^2} + \|\phi\|_{L^2} \lesssim \|\mathring{D}\phi\|_{L^2} + \|\phi\|_{L^2}.$$

This implies (1). The proof of (2) is similar. ■

Proof of Proposition 3.17. By Lemma 3.18, cut-off is an isomorphism of Hilbert spaces. To prove that cut-off is a symplectomorphism, let $\phi, \psi \in \operatorname{dom}(D_{\max})$ and set

$$v \coloneqq \sum_{i=1}^n \langle\gamma(e_i)\phi, \psi\rangle e_i \quad\text{and}\quad \mathring{v} \coloneqq \sum_{i=1}^n \langle\mathring{\gamma}(\mathring{e}_i)\chi(r)\cdot\phi, \chi(r)\cdot\psi\rangle\mathring{e}_i$$

with $(e_1,\dots,e_n)$ and $(\mathring{e}_1,\dots,\mathring{e}_n)$ denoting local g– and $\mathring{g}$–orthonormal frames respectively.

Assume, without loss of generality, that $\operatorname{supp}(\phi) \cup \operatorname{supp}(\psi) \subset (\chi \circ r)^{-1}(1) \subset U \backslash Z$. With η_ε as in the proof of Lemma 3.18,

$$\begin{aligned}(G - \text{cut-off}^* \mathring{G})([\phi] \wedge [\psi]) &= \int_{U \backslash Z} \operatorname{div}_g(v) \cdot \mathrm{vol}_g - \operatorname{div}_{\mathring{g}}(\mathring{v}) \cdot \mathrm{vol}_{\mathring{g}} \\ &= \lim_{\varepsilon \downarrow 0} \int_{U \backslash Z} \eta_\varepsilon \cdot \big(\operatorname{div}_g(v) \cdot \mathrm{vol}_g - \operatorname{div}_{\mathring{g}}(\mathring{v}) \cdot \mathrm{vol}_{\mathring{g}}\big) \\ &= -\lim_{\varepsilon \downarrow 0} \int_{U \backslash Z} \mathrm{d}\eta_\varepsilon \wedge \big(i_v \mathrm{vol}_g - i_{\mathring{v}} \mathrm{vol}_{\mathring{g}}\big).\end{aligned}$$

Since $r|\mathrm{d}\eta_\varepsilon| \lesssim 1$,

$$\big|\mathrm{d}\eta_\varepsilon \wedge \big(i_v \mathrm{vol}_g - i_{\mathring{v}} \mathrm{vol}_{\mathring{g}}\big)\big| \lesssim |\phi||\psi|.$$

Therefore,

$$|(G - \text{cut-off}^* \mathring{G})([\phi] \wedge [\psi])| \lesssim \lim_{\varepsilon \downarrow 0} \int_{\operatorname{supp} \mathrm{d}\eta_\varepsilon} |\phi||\psi| \, \mathrm{vol}_{\mathring{g}} = 0. \qquad \blacksquare$$

3.4 Spectral decomposition

This subsection decomposes $(\mathring{\mathbf{H}}_0, \mathring{G})$ into concretely understandable summands.

Definition 3.19 ($\mathfrak{l}$ determines NZ^λ). The ramified Euclidean line bundle $\mathfrak{l}$ determines the following:

(1) The 2π–periodic vector field ∂_α generating $F \circlearrowleft \mathrm{U}(1)$ uniquely lifts along

$$\rho \colon \ \tilde{F} \coloneqq \{\ell \in \underline{\mathfrak{l}} : |\ell| = 1\} \longrightarrow F$$

to a 4π–periodic vector field $\frac{1}{2}\partial_\beta$. The 2π–periodic vector field ∂_β generates $\tilde{F} \circlearrowleft \mathrm{U}(1)$ with respect to which $\tilde{\pi} \colon \ \tilde{F} \longrightarrow Z$ is a $\mathrm{U}(1)$–principal bundle.

(2) Let $\lambda \in \frac{1}{2}\mathbf{Z}$. The Hermitian line bundle

$$NZ^\lambda \coloneqq \tilde{F} \times_{\mathrm{U}(1)} \mathbf{C}$$

arises from $\tilde{F}$ via the representation $\mathrm{U}(1) \circlearrowleft \mathbf{C}$ of weight 2λ. The Levi-Civita connection on F induces a connection on $\tilde{F}$ and, therefore, a unitary covariant derivative ∇^λ on NZ^λ. ●

Remark 3.20. By construction $(NZ^1, \nabla^1) \cong (NZ, \nabla^{\mathrm{LC}})$ and for every $\lambda, \mu \in \frac{1}{2}\mathbf{Z}$

$$(NZ^\lambda, \nabla^\lambda) \otimes_{\mathbf{C}} (NZ^\mu, \nabla^\mu) \cong (NZ^{\lambda+\mu}, \nabla^{\lambda+\mu}). \qquad \clubsuit$$

Proposition 3.21. *For every $\lambda \in \mathbf{Z} - 1/2$ there is an isomorphism*

$$P_\lambda \colon \ \pi^*(NZ^\lambda, \nabla^\lambda) \cong (\underline{\mathfrak{l}} \otimes \mathbf{C}, \nabla_{\underline{\mathfrak{l}} \otimes \mathbf{C}} + i\lambda\theta)$$

of Hermitian line bundles with unitary connections.

Proof. Consider the $\mathrm{U}(1)$–principal bundle $\tilde{F} \times_{\{\pm 1\}} \mathrm{U}(1) \to F$ obtained by extending the $\{\pm 1\}$–principal bundle $\tilde{F} \to F$ along the inclusion $\iota\colon \{\pm 1\} \hookrightarrow \mathrm{U}(1)$. The $\mathrm{U}(1)$–principal bundles $\tilde{F} \times_{\{\pm 1\}} \mathrm{U}(1) \to F$ and $\pi^* \tilde{F} \to F$ are isomorphic via $[f, z] \mapsto [f, \rho(f \cdot z)]$.

Let $\lambda \in \mathbf{Z} - 1/2$. The representation $\mathrm{U}(1) \circlearrowright \mathbf{C}$ of weight 2λ restricts to the usual representation $\{\pm 1\} \circlearrowright \mathbf{C}$ along ι. Therefore, $\pi^* NZ^\lambda$ and $\underline{I}$ both arise from the representation of weight 2λ. Hence, they are isomorphic as Hermitian line bundles.

The Levi-Civita connection $i\theta$ on $F \to \mathbf{Z}$ induces the connection $\frac{i}{2}\rho^*\theta$ on $\tilde{F} \to Z$. Therefore, the induced connection on $\tilde{F} \times_{\{\pm 1\}} \mathrm{U}(1) \cong \pi^* F$ is (the descend of) $\frac{i}{2}\rho^*\theta + \mu_{\mathrm{U}(1)}$. Here $\mu_{\mathrm{U}(1)} \in \mathrm{U}(1)$ denotes the Maurer–Cartan form on $\mathrm{U}(1)$. The connection on $\tilde{F} \times_{\{\pm 1\}} \mathrm{U}(1)$ induced by the flat connection on $\tilde{F} \to F$ is (the descend) of $\mu_{\mathrm{U}(1)}$. This implies the assertion about the covariant derivatives. ■

Proposition 3.22 (Spectral decomposition of $L^2\Gamma(F, \underline{S} \otimes \underline{I})$). *For every $\lambda \in \mathbf{Z} - 1/2$ and $\mu \in \mathbf{R}$ set*

$$E_{\lambda,\mu} \coloneqq \{\phi \in L^2\Gamma(F, \underline{S} \otimes \underline{I}) : I\mathring{\nabla}_{\partial_\alpha}\phi = \lambda\phi, D_Z\phi = \mu\phi\} \quad \textit{and}$$
$$\check{E}_{\lambda,\mu} \coloneqq \{\check{\phi} \in L^2\Gamma(Z, S|_Z \otimes_{\mathbf{C}} NZ^\lambda) : D_{S|_Z \otimes_{\mathbf{C}} NZ^\lambda}\check{\phi} = \mu\check{\phi}\}.$$

Here D_Z is as in Remark 3.13 and $D_{S|_Z \otimes_{\mathbf{C}} NZ^\lambda}$ arises from Definition 3.9 and twisting by $(NZ^\lambda, \nabla^\lambda)$; moreover: the tensor product is with respect to the complex structure I on $S|_Z$. The following hold:

(1) *For every $\lambda \in \mathbf{Z} - 1/2$ and $\mu \in \mathbf{R}$, P_λ induces an isomorphism*

$$\pi^* \check{E}_{\lambda,\mu} \cong E_{\lambda,\mu}.$$

(2) *The subset*

$$\sigma \coloneqq \left\{(\lambda, \mu) \in (\mathbf{Z} - 1/2) \times \mathbf{R} : E_{\lambda,\mu} \neq 0\right\}$$

is discrete. Moreover, for every $(\lambda, \mu) \in \sigma$, $E_{\lambda,\mu}$ is finite-dimensional.

(3) *For every $(\lambda, \mu) \in \sigma$*

$$JE_{\lambda,\mu} = E_{-(\lambda+1),-\mu}.$$

(4) *The Hilbert space $L^2\Gamma(F, \underline{S} \otimes \underline{I})$ decomposes as a (Hilbert space) direct sum*

$$L^2\Gamma(F, \underline{S} \otimes \underline{I}) = \bigoplus_{(\lambda,\mu) \in \sigma} E_{\lambda,\mu}.$$

Proof. By Fourier analysis, the Hilbert space $L^2\Gamma(F, \underline{S} \otimes \underline{I})$ decomposes as a direct sum

$$L^2\Gamma(F, \underline{S} \otimes \underline{I}) = \bigoplus_{\lambda \in \mathbf{Z} - 1/2} E_\lambda \quad \text{with} \quad E_\lambda \coloneqq \{\phi \in L^2\Gamma(F, \underline{S} \otimes \underline{I}) : I\mathring{\nabla}_{\partial_\alpha}\phi = \lambda\phi\}$$

By Proposition 3.21, P_λ induces an isomorphism $\pi^* L^2\Gamma(Z, S|_Z \otimes_{\mathbf{C}} NZ^\lambda) \cong E_\lambda$. By the spectral theory of Dirac operators, for every $\lambda \in \mathbf{Z} - 1/2$, $\operatorname{spec}(D_{S|_Z \otimes_{\mathbf{C}} NZ^\lambda}) \subset \mathbf{R}$ is discrete and the Hilbert space $L^2\Gamma(Z, S|_Z \otimes_{\mathbf{C}} NZ^\lambda)$ decomposes as a direct sum finite-dimensional eigenspaces $\check{E}_{\lambda,\mu}$ of $D_{S|_Z \otimes_{\mathbf{C}} NZ^\lambda}$. This proves (1), (2), and (4).

(3) holds because J and D_Z anti-commute and $I\mathring{\nabla}_{\partial_\alpha} J = -J(I\mathring{\nabla}_{\partial_\alpha} + 1)$ since $\mathring{\nabla}_{\partial_\alpha}\partial_r = \partial_\alpha$. ■

Since $\mathrm{vol}_{\mathring{g}} = \mathrm{d}r \wedge r\theta \wedge \mathrm{vol}_{g|_Z}$, by Fubini's theorem and Proposition 3.22,

$$L^2\Gamma(U\backslash Z, \mathring{S} \otimes \mathring{\mathrm{I}}) = L^2((0,1), r\mathrm{d}r; L^2\Gamma(F, \underline{S} \otimes \underline{\mathrm{I}})) = \bigoplus_{\lambda \in \mathbf{Z}-1/2} \bigoplus_{\mu \in \sigma_\lambda} L^2((0,1), r\mathrm{d}r; E_{\lambda,\mu});$$

moreover, $D_{\max}$ decomposes as follows.

Definition 3.23. *Choose* a fundamental domain $\breve{\sigma} \subset \sigma$ for the involution $(\lambda, \mu) \mapsto (-(\lambda+1), -\mu)$. *Choose* a real subspace $E^{\mathbf{R}}_{-1/2,0} \subset E_{-1/2,0}$ with respect to J. For every $(\lambda, \mu) \in \breve{\sigma}$ set

$$V_{\lambda,\mu} \coloneqq \begin{cases} E^{\mathbf{R}}_{-1/2,0} \oplus JE^{\mathbf{R}}_{-1/2,0} & \text{if } (\lambda,\mu) = (-1/2, 0) \\ E_{\lambda,\mu} \oplus E_{-(\lambda+1),-\mu} & \text{otherwise;} \end{cases}$$

moreover, define $\mathring{D}^{\lambda,\mu} \colon H^1_{\mathrm{loc}}((0,1); V_{\lambda,\mu}) \longrightarrow L^2_{\mathrm{loc}}((0,1); V_{\lambda,\mu})$ by

$$\mathring{D}^{\lambda,\mu} \coloneqq \begin{pmatrix} \mu & J(\partial_r + \frac{\lambda+1}{r}) \\ J(\partial_r - \frac{\lambda}{r}) & -\mu \end{pmatrix}$$

and set

$$\mathrm{dom}(\mathring{D}^{\lambda,\mu}_{\max}) \coloneqq \left\{ \phi \in H^1_{\mathrm{loc}}((0,1); V_{\lambda,\mu}) : \phi, \mathring{D}\phi \in L^2((0,1), r\mathrm{d}r; V_{\lambda,\mu}) \right\}.$$ ●

Remark 3.24. The purpose of the artificial decomposition of $E_{-1/2,0}$ is to avoid a case distinction in the definition of $\mathring{D}^{\lambda,\mu}$. ♣

Proposition 3.25 (Spectral decomposition of $\mathrm{dom}(\mathring{D}_{\max})$). *The following hold:*

(1) *The Hilbert space* $\mathrm{dom}(\mathring{D}_{\max})$ *decomposes as a (Hilbert space) direct sum*

$$\mathrm{dom}(\mathring{D}_{\max}) = \bigoplus_{(\lambda,\mu) \in \breve{\sigma}} \mathrm{dom}(\mathring{D}^{\lambda,\mu}_{\max}).$$

(2) *The model operator* $\mathring{D}$ *decomposes as*

$$\mathring{D} = \bigoplus_{(\lambda,\mu) \in \breve{\sigma}} \mathring{D}^{\lambda,\mu}.$$

Proof. This is an immediate consequence of Remark 3.13 and Proposition 3.22. ■

Remark 3.26. The ordinary differential equation $\mathring{D}^{\lambda,\mu}\phi = \psi$ can be solved explicitly in terms of modified Bessel functions of the second kind or using the Hankel transform. However, none of this is necessary for the purpose of this article. ♣

Finally, here is the desired decomposition of $(\mathring{\check{\mathbf{H}}}_0, \mathring{G})$.

Corollary 3.27 (Spectral decomposition of $(\mathring{\check{\mathbf{H}}}_0, \mathring{G})$). *The symplectic Hilbert space* $(\mathring{\check{\mathbf{H}}}_0, \mathring{G})$ *decomposes as a (Hilbert space) direct sum*

$$(\mathring{\check{\mathbf{H}}}_0, \mathring{G}) = \bigoplus_{(\lambda,\mu) \in \breve{\sigma}} (\mathring{\check{\mathbf{H}}}^{\lambda,\mu}_0, \mathring{G}_{\lambda,\mu})$$

with

$$\mathring{\check{\mathbf{H}}}_0^{\lambda,\mu} := \frac{(\chi \circ r) \cdot \operatorname{dom}(\mathring{D}^{\lambda,\mu}_{\max}) + \operatorname{dom}(\mathring{D}_{\min})}{\operatorname{dom}(\mathring{D}_{\min})} \quad \textit{and}$$

$$\mathring{G}_{\lambda,\mu}([\phi] \wedge [\psi]) := \langle \mathring{D}^{\lambda,\mu}\phi, \psi\rangle_{L^2} - \langle \phi, \mathring{D}^{\lambda,\mu}\psi\rangle_{L^2}. \qquad \blacksquare$$

3.5 Leading order terms

This subsection determines $(\mathring{\check{\mathbf{H}}}_0^{\lambda,\mu}, \mathring{G}_{\lambda,\mu})$ based on the following observation.

Lemma 3.28 (Leading order terms; cf. [BS88, Lemma 2.1; DW24, Lemma 3.50]). *Let* $\lambda \in \mathbf{R}$. *Let* $\phi \in H^1_{\mathrm{loc}}((0,1))$ *with* $\phi, (\partial_r - \lambda/r)\phi \in L^2((0,1), r\mathrm{d}r)$. *The following hold:*

1. *If* $\lambda \in (-1,0)$, *then there is a unique* $a \in \mathbf{R}$ *such that* $\lim_{r\downarrow 0} \phi(r) - ar^\lambda = 0$.
2. *If* $\lambda = 0$, *then* $\phi(r) \lesssim_\phi |\log(r)|^{1/2}$.
3. *If* $\lambda \neq [-1,0)$, *then* $\lim_{r\downarrow 0}\phi(r) = 0$.
4. *If* $\lim_{r\downarrow 0}\phi(r) = \lim_{r\uparrow 1}\phi(r) = 0$, *then*

$$\int_0^1 \left(|\partial_r\phi|^2 + \frac{\lambda^2}{r^2}|\phi|^2\right) r\mathrm{d}r = \int_0^1 |(\partial_r - \lambda/r)\phi|^2 r\mathrm{d}r.$$

Proof. The proof is almost identical to that of [DW24, Lemma 3.50] and is repeated here only for the readers' convenience.

Evidently, $(\partial_r - \lambda/r)r^\lambda = 0$ and $r^\lambda \in L^2((0,1), r\mathrm{d}r)$ if and only if $\lambda > -1$. Set $\psi := (\partial_r - \lambda/r)\phi$. By variation of parameters, there is a unique $a \in \mathbf{R}$ such that

$$\tilde{\phi}(r) := \phi(r) - ar^\lambda = \begin{cases} r^\lambda \int_0^r s^{-(\lambda+1)}\psi(s)\,\mathrm{s}\mathrm{d}s & \text{if } \lambda < 0 \\ -r^\lambda \int_r^1 s^{-(\lambda+1)}\psi(s)\,\mathrm{s}\mathrm{d}s & \text{if } \lambda \geqslant 0. \end{cases}$$

Of course, if $\lambda \leqslant -1$, then $a = 0$.

If $\lambda < 0$, then, by Cauchy–Schwarz and monotone convergence,

$$|\tilde{\phi}(r)|^2 \leqslant \frac{1}{2|\lambda|}\int_0^r |\psi(s)|^2\,\mathrm{s}\mathrm{d}s = o(1) \quad \text{as} \quad r \downarrow 0.$$

If $\lambda = 0$, then

$$|\tilde{\phi}(r)|^2 \leqslant |\log(r)| \int_r^1 |\psi(s)|^2\,\mathrm{s}\mathrm{d}s = O(|\log(r)|) \quad \text{as} \quad r \downarrow 0.$$

If $\lambda > 0$, then, by Cauchy–Schwarz, for $r \leqslant \varepsilon \leqslant 1$

$$|\tilde{\phi}(r)|^2 \leqslant \frac{1}{\lambda}\int_0^\varepsilon |\psi(s)|^2\,\mathrm{s}\mathrm{d}s + \frac{(r/\varepsilon)^{2\lambda}}{\lambda}\int_\varepsilon^1 |\psi(s)|^2\,\mathrm{s}\mathrm{d}s =: \mathrm{I}(\varepsilon) + \mathrm{II}(r,\varepsilon).$$

By monotone convergence, $\lim_{\varepsilon\downarrow 0}\mathrm{I}(\varepsilon) = 0$. Evidently, $\lim_{r\downarrow 0}\mathrm{II}(r,\varepsilon) = 0$. Therefore, $\tilde{\phi}(r) = o(1)$ as $r \downarrow 0$. These observations imply (1), (2), and (3).

(4) is a consequence of

$$\int_0^1 |(\partial_r - \lambda/r)\phi|^2 r\mathrm{d}r = \int_0^1 \left(|\partial_r\phi|^2 + \frac{\lambda^2}{r^2}|\phi|^2\right) r\mathrm{d}r - \lambda\int_0^1 \partial_r|\phi|^2\mathrm{d}r. \qquad \blacksquare$$

Corollary 3.29 (Identification of $\mathring{\check{\mathbf{H}}}_0^{\lambda,\mu}$ for $\lambda \neq -1/2$). *For every* $(\lambda,\mu) \in \check{\sigma}$ *with* $\lambda \neq -1/2$

$$\mathring{\check{\mathbf{H}}}_0^{\lambda,\mu} = 0. \qquad \blacksquare$$

Definition 3.30. For $(-1/2,\mu) \in \check{\sigma}$ define the **residue map** $\mathrm{res}_\mu\colon\ \mathrm{dom}(\mathring{D}_{\max}^{-1/2,\mu}) \longrightarrow V_{-1/2,\mu}$ by

$$\phi - r^{-1/2}\cdot \mathrm{res}_\mu(\phi) \in \mathrm{dom}(\mathring{D}_{\min}),$$

and the symplectic form $\check{\Omega}_\mu \in \mathrm{Hom}(\Lambda^2 V_{-1/2,\mu}, \mathbf{R})$ by

$$\check{\Omega}_\mu(\phi\wedge\psi) \coloneqq -\langle J\phi,\psi\rangle. \qquad \bullet$$

Proposition 3.31 (Identification of $\mathring{\check{\mathbf{H}}}_0^{-1/2,\mu}$: symplectic structure). *For every* $(-1/2,\mu) \in \check{\sigma}$ *the residue map induces an isomorphism*

$$\mathrm{res}_\mu\colon\ (\mathring{\check{\mathbf{H}}}_0^{-1/2,\mu}, \mathring{G}^{-1/2,\mu}) \cong (V_{-1/2,\mu}, \check{\Omega}_\mu).$$

Proof. For $\phi,\psi \in \mathrm{dom}(\mathring{D}_{\max}^{-1/2,\mu})$, by direct computation using $\partial_r + \frac{1}{2r} = r^{-1/2}\partial_r r^{1/2}$,

$$\begin{aligned}\mathring{G}^{-1/2,\mu}([\phi]\wedge[\psi]) &= \int_0^1 \left(\langle J(\partial_r + \tfrac{1}{2r})\phi,\psi\rangle - \langle\phi, J(\partial_r + \tfrac{1}{2r})\psi\rangle\right) r\mathrm{d}r \\ &= \int_0^1 \partial_r\langle Jr^{1/2}\phi, r^{1/2}\psi\rangle\,\mathrm{d}r \\ &= -\langle J\,\mathrm{res}_\mu([\phi]), \mathrm{res}_\mu([\psi])\rangle = \mathrm{res}_\mu^*\,\check{\Omega}_\mu([\phi]\wedge[\psi]).\end{aligned}$$

This together with Lemma 3.28 immediately implies the assertion. ■

Although res_μ is an isomorphism, the norms on $\mathring{\check{\mathbf{H}}}_0^{-1/2,\mu}$ and $V_{-1/2,\mu}$ are *not uniformly* equivalent. The following discussion rectifies this.

Definition 3.32. Let $(-1/2,\mu) \in \check{\sigma}$.

(1) Define the **branching locus operator** $A_\mu\colon\ V_{-1/2,\mu} \longrightarrow V_{-1/2,\mu}$ by

$$A_\mu \coloneqq -J\mathring{D}^{-1/2,\mu} - \partial_r - \tfrac{1}{2r} = \begin{pmatrix} 0 & J\mu \\ -J\mu & 0\end{pmatrix}.$$

(2) Define the norm $\|-\|_{\check{H}}\colon\ V_{-1/2,\mu} \longrightarrow [0,\infty)$ by

$$\|v\|_{\check{H}}^2 \coloneqq (1+|\mu|)\cdot\|\mathbf{1}_{(-\infty,0)}(A_\mu)\phi\|^2 + (1+|\mu|)^{-1}\cdot\|\mathbf{1}_{[0,\infty)}(A_\mu)\phi\|^2.$$

Here $\mathbf{1}_{(-\infty,0)}(A_\mu)$ and $\mathbf{1}_{[0,\infty)}(A_\mu)$ denote the orthogonal projection to the negative and non-negative eigenspaces of A_μ respectively.

(3) Define the norm $\|-\|_{H^{-1/2}}\colon V_{-1/2,\mu} \longrightarrow [0,\infty)$ by

$$\|v\|_{H^{-1/2}}^2 \coloneqq (1+|\mu|)^{-1} \cdot \|v\|^2. \qquad \bullet$$

Proposition 3.33 (Identification of $\mathring{\check{\mathbf{H}}}_0^{-1/2,\mu}$: uniform norms). *For every* $(-1/2,\mu) \in \check{\sigma}$

$$\|\mathrm{res}_\mu([\phi])\|_{\check{H}} \asymp \|[\phi]\|_{\mathring{\check{\mathbf{H}}}}.$$

The proof uses the following right inverse of $\mathrm{res}_\mu \circ [\cdot]$.

Definition 3.34. For $(-1/2,\mu) \in \check{\sigma}$ define the **extension map** $\mathrm{ext}_\mu\colon V_{-1/2,\mu} \longrightarrow \mathrm{dom}(\mathring{D}_{\max}^{-1/2,\mu})$ by

$$\mathrm{ext}_\mu(v) \coloneqq r^{-1/2}e^{-|\mu|r} \cdot v. \qquad \bullet$$

Evidently, ext_μ lifts the inverse of $\mathrm{res}_\mu\colon \mathring{\check{\mathbf{H}}}_0^{-1/2,\mu} \cong V_{-1/2,\mu}$. Therefore, Proposition 3.33 is an immediate consequence of the following.

Lemma 3.35 (Uniform estimates for res_μ and ext_μ). *Let* $(-1/2,\mu) \in \check{\sigma}$. *The following hold:*

(1) *For every* $\phi \in \mathrm{dom}(\mathring{D}_{\max}^{-1/2,\mu})$

$$\|\mathrm{res}_\mu([\phi])\|_{\check{H}} \lesssim \|\phi\|_{\mathring{D}}.$$

(2) *For every* $v \in V_{-1/2,\mu}$

$$\|\mathrm{ext}_\mu(v)\|_{L^2} \lesssim \|v\|_{H^{-1/2}} \quad \textit{and} \quad \|\mathrm{ext}_\mu(v)\|_{\mathring{D}} \lesssim \|v\|_{\check{H}}.$$

Proof. Evidently, $v \coloneqq \mathrm{res}_\mu([\phi])$ satisfies

$$v = -\int_0^1 \partial_r(r^{1/2}e^{-|\mu|r}\phi)\,\mathrm{d}r = -\int_0^1 r^{1/2}(-J\mathring{D}^{-1/2,\mu} - A_\mu - |\mu|)\phi \cdot e^{-|\mu|r}\,\mathrm{d}r.$$

Therefore, by Cauchy–Schwarz,

$$\begin{aligned}\|\mathbf{1}_{(-\infty,0)}(A_\mu)v\|^2 &\lesssim \int_0^1 \|\mathring{D}^{-1/2,\mu}\mathbf{1}_{(-\infty,0)}(A_\mu)\phi\|^2\,r\mathrm{d}r \cdot \int_0^1 e^{-2|\mu|r}\,\mathrm{d}r \\ &\lesssim \int_0^1 e^{-2|\mu|r}\,\mathrm{d}r \cdot \|\mathbf{1}_{(-\infty,0)}(A_\mu)\phi\|_{\mathring{D}}^2\end{aligned}$$

and

$$\begin{aligned}\|\mathbf{1}_{[0,\infty)}(A_\mu)v\|^2 &\lesssim \int_0^1 \Big(\|\mathring{D}^{-1/2,\mu}\mathbf{1}_{[0,\infty)}(A_\mu)\phi\|^2 + |\mu|^2\|\mathbf{1}_{[0,\infty)}(A_\mu)\phi\|^2\Big)\,r\mathrm{d}r \cdot \int_0^1 e^{-2|\mu|r}\,\mathrm{d}r \\ &\lesssim (1+|\mu|)^2 \cdot \int_0^1 e^{-2|\mu|r}\,\mathrm{d}r \cdot \|\mathbf{1}_{[0,\infty)}(A_\mu)\phi\|_{\mathring{D}}^2.\end{aligned}$$

The estimate in (1) follows because

$$\int_0^1 e^{-2|\mu|r}\,\mathrm{d}r \asymp (1+|\mu|)^{-1} \quad \text{and} \quad \|\mathbf{1}_{(-\infty,0)}(A_\mu)\phi\|_{\mathring{D}}^2 + \|\mathbf{1}_{[0,\infty)}(A_\mu)\phi\|_{\mathring{D}}^2 = \|\phi\|_{\mathring{D}}^2.$$

To prove (2), observe that

$$\|\mathrm{ext}_\mu(v)\|_{L^2}^2 \lesssim \int_0^1 r^{-1}e^{-2|\mu|r}\|v\|^2\, r\mathrm{d}r \lesssim (1+|\mu|)^{-1}\cdot\|v\|^2$$

and

$$\begin{aligned}\|\mathring{D}^{-1/2,\mu}\,\mathrm{ext}_\mu(v)\|_{L^2}^2 &= \int_0^1 \big(r^{-1/2}e^{-|\mu|r}\big)^2\|(-|\mu|+A_\mu)v\|^2\, r\mathrm{d}r \\ &\lesssim \int_0^1 \big(r^{-1/2}|\mu|e^{-|\mu|r}\big)^2\|\mathbf{1}_{(-\infty,0]}(A_\mu)v\|^2\, r\mathrm{d}r \\ &\lesssim \mu^2(1+|\mu|)^{-1}\cdot\|v\|^2. \end{aligned}$$ ■

3.6 Assembly of the residue map

This subsection (re)assembles the summands of the decomposition Corollary 3.27 identified in Section 3.5 in a more geometric fashion.

Definition 3.36.

(1) The **residue bundle** is

$$\check{S} \coloneqq S|_Z \otimes_{\mathbf{C}} NZ^{-1/2}.$$

As a consequence of Proposition 3.22 (3) (or by direct inspection), $\check{S}$ inherits the quaternionic structure $I, J, K = IJ$ from $\underline{S}$. Define the symplectic form $\check{\Omega} \in \Gamma(Z, \mathrm{Hom}(\Lambda^2\check{S}, \mathbf{R}))$ by

$$\check{\Omega} \coloneqq -2\pi\langle J-, -\rangle.$$

(2) The **branching locus operator** $A\colon \Gamma(Z,\check{S}) \longrightarrow \Gamma(Z,\check{S})$ is defined by

$$A \coloneqq -JD_{\check{S}}$$

with $D_{\check{S}} \coloneqq D_{S|_Z\otimes_C NZ^{-1/2}}$ as in Proposition 3.22. Since J and $D_{\check{S}}$ anti-commute, A is (formally) self-adjoint.

(3) Denote by $\mathbf{1}_{(-\infty,0)}(A)$ and $\mathbf{1}_{[0,\infty)}(A)$ the orthogonal projection to the negative and non-negative eigenspaces of A respectively. Define the norm $\|-\|_{\check{H}}\colon \Gamma(Z,\check{S}) \longrightarrow [0,\infty)$ by

$$\|\phi\|_{\check{H}} \coloneqq \|\mathbf{1}_{(-\infty,0)}(A)\phi\|_{H^{1/2}} + \|\mathbf{1}_{[0,\infty)}(A)\phi\|_{H^{-1/2}}$$

and denote by $\check{H}\Gamma(Z,\check{S})$ the completions of $\Gamma(Z,\check{S})$ with respect to $\|-\|_{\check{H}}$. •

Proposition 3.37. $\check{\Omega}$ *extends to a symplectic structure* $\check{\Omega} \in \mathscr{L}(\Lambda^2\check{H}\Gamma(Z,\check{S}), \mathbf{R})$*; moreover: the inclusion* $V_{-1/2,\mu} \hookrightarrow \Gamma(Z,\check{S})$ *assemble into an isomorphism of symplectic Hilbert spaces*

$$\bigoplus_{(-1/2,\mu)\in\check{\sigma}} (V_{-1/2,\mu}, \|-\|_{\check{H}}; \check{\Omega}_\mu) \cong (\check{H}\Gamma(Z,\check{S}); \check{\Omega}).$$

Proof. This is an immediate consequence of Proposition 3.22, Proposition 3.31, and Proposition 3.33. The (possibly mysterious) factor 2π arises because $\check{\Omega}_\mu$ is defined using the L^2 inner product on F instead of Z and $\mathrm{vol}(F) = 2\pi\mathrm{vol}(Z)$. ■

Definition 3.38.

(1) The **residue map** $\mathrm{res}\colon \check{\mathbf{H}} \to \check{H}\Gamma(Z, \check{S})$ obtained as the composition of the following maps

$$\check{\mathbf{H}} \xrightarrow{\text{cut-off}} \mathring{\check{\mathbf{H}}}_0 = \bigoplus_{(-1/2,\mu)\in\check{\sigma}} \mathring{\check{\mathbf{H}}}_0^{-1/2,\mu} \xrightarrow{(\mathrm{res}_\mu)} \bigoplus_{(-1/2,\mu)\in\check{\sigma}} (V_{-1/2,\mu}, \|-\|_{\check{H}}) \cong \check{H}\Gamma(Z, \check{S}).$$

(2) The **extension map** $\mathrm{ext}\colon \check{H}\Gamma(Z, \check{S}) \to \mathrm{dom}(D_{\max})$ is obtained as the composition of the following maps

$$\check{H}\Gamma(Z, \check{S}) \cong \bigoplus_{(-1/2,\mu)\in\check{\sigma}} (V_{-1/2,\mu}, \|-\|_{\check{H}}) \xrightarrow{(\mathrm{ext}_\mu)} \bigoplus_{(-1/2,\mu)\in\check{\sigma}} \mathrm{dom}(\mathring{D}_{\max}^{-1/2,\mu})$$
$$\hookrightarrow \mathrm{dom}(\mathring{D}_{\max}) \xrightarrow{\chi\circ r} \mathrm{dom}(D_{\max}). \qquad \bullet$$

Theorem 3.39. *The following hold:*

(1) (a) *The residue map is an isomorphism of symplectic Hilbert spaces*

$$\mathrm{res}\colon (\check{\mathbf{H}}, G) \cong (\check{H}\Gamma(Z, \check{S}), \check{\Omega}).$$

(b) *The subspace* $r^{-1/2}\Gamma(\hat{X}, \hat{S} \otimes \hat{\mathfrak{l}}) \cap \mathrm{dom}(D_{\max})$ *is dense in* $\mathrm{dom}(D_{\max})$*; hence: the residue map is uniquely determined by*

$$\pi^* \,\mathrm{res}[r^{-1/2}\phi] = \phi|_{\partial\hat{X}}$$

for every $r^{-1/2}\phi \in r^{-1/2}\Gamma(\hat{X}, \hat{S} \otimes \hat{\mathfrak{l}}) \cap \mathrm{dom}(D_{\max})$.

(2) (a) *The extension map* $\mathrm{ext}\colon \check{H}\Gamma(Z, \check{S}) \to \mathrm{dom}(D_{\max})$ *is a right-inverse of* $\mathrm{res} \circ [-]$.

(b) *The extension map extends to a bounded linear map*

$$\mathrm{ext}\colon H^{-1/2}\Gamma(Z, \check{S}) \to L^2\Gamma(X\backslash Z, S \otimes \mathfrak{l}).$$

Proof. (1.a) is an immediate consequence of Proposition 3.17, Corollary 3.27, Proposition 3.31, Proposition 3.33, and Proposition 3.37.

(1.b) is a consequence of Lemma 3.18 and Proposition 3.25.

(2.a) holds by construction and (2.b) follows from Lemma 3.35 (2). ■

Remark 3.40. If ε is a chirality operator, then S orthogonally decomposes as $\check{S} = \check{S}^+ \oplus \check{S}^-$, $\check{S}^\pm \subset \check{S}$ are Lagrangian subbundles, A preserves this splitting, $\check{H}\Gamma(Z, \check{S})$ orthogonally decomposes as $\check{H}\Gamma(Z, \check{S}) = \check{H}\Gamma(Z, \check{S}^+) \oplus \check{H}\Gamma(Z, \check{S}^-)$, and the residue map restricts to isomorphism

$$\mathrm{res}\colon \check{\mathbf{H}}^\pm \cong \check{H}\Gamma(Z, \check{S}^\pm). \qquad ♣$$

3.7 Spectral and local residue conditions

Theorem 3.39 makes it possible to define a wider variety of residue conditions than those considered in Section 2. Here are some examples.

Example 3.41. The **APS residue condition** is defined by

$$B_{\mathrm{APS}} \coloneqq \mathbf{1}_{(-\infty,0)}(A)H^{1/2}\Gamma(Z,\check{S}) \subset \check{H}\Gamma(Z,\check{S});$$

cf. Atiyah, Patodi, and Singer [APS75, (2.3)]. ♠

Proposition 3.42 (Criterion for left semi-Fredholmness). *Let $B \subset \check{H}\Gamma(Z,\check{S})$ be a residue condition. If $B \hookrightarrow H^{-1/2}\Gamma(Z,\check{S})$ is compact, then* $\mathrm{dom}(D_B) \hookrightarrow L^2\Gamma(X\backslash Z, S\otimes \mathfrak{l})$ *is compact and D_B is left semi-Fredholm.*

Proof. By Lemma 2.4 (2), and the assumption, the composition

$$\begin{aligned}\mathrm{dom}(D_B) &\to H^1\Gamma(X\backslash Z, S\otimes \mathfrak{l}) \oplus B \hookrightarrow L^2\Gamma(X\backslash Z, S\otimes \mathfrak{l}) \oplus H^{-1/2}\Gamma(Z,\check{S})\\ \phi &\mapsto (\phi - \mathrm{ext}\,\mathrm{res}[\phi], \mathrm{res}[\phi])\end{aligned}$$

is compact. Therefore, by Theorem 3.39 (2.b), $\mathrm{dom}(D_B) \hookrightarrow L^2\Gamma(X\backslash Z, S\otimes \mathfrak{l})$ is compact. Since for every $\phi \in \mathrm{dom}(D_B)$

$$\|\phi\|_D \lesssim \|D\phi\|_{L^2} + \|\phi\|_{L^2},$$

D_B is left semi-Fredholm. ■

Example 3.43. Since $B_{\mathrm{APS}}^G = B_{\mathrm{APS}} \oplus \ker A$, by Proposition 2.28 and Proposition 3.42, $D_{B_{\mathrm{APS}}}$ is Fredholm of index $-\frac{1}{2}\dim\ker A$. In particular, $\dim\ker A$ is even and inherits a symplectic structure from G. If $L \subset \ker A$ is Lagrangian, then $B_{\mathrm{APS}} \oplus L \hookrightarrow H^{-1/2}\Gamma(Z,\check{S})$ is compact and Lagrangian. In particular, Proposition 2.22 applies. ♠

Definition 3.44. Let $V \subset \check{S}$ be a subbundle. The **local residue condition** associated with V is

$$B_V \coloneqq \check{H}\Gamma(Z,V) \subset \check{H}\Gamma(Z,\check{S}). \qquad \bullet$$

Proposition 3.45. *Let $V \subset \check{S}$ be a subbundle. If $V^{\check{\Omega}}$ denotes the symplectic complement of $V \subset \check{S}$, then*

$$B_V^G = B_{V^{\check{\Omega}}}.$$

Proof. This is an immediate consequence of Proposition 2.17 and Theorem 3.39. ■

Example 3.46. Consider the Dirac bundle (S,γ,∇) corresponding to the Hodge–de Rham operator $\mathrm{d}+\mathrm{d}^*$; that is: $S \coloneqq \Lambda T^*X$ with $\gamma(\xi)\phi \coloneqq \xi\wedge\phi - i_{\xi^\sharp}\phi$. Decompose

$$\check{S} = \check{S}_N \oplus \check{S}_D$$

with $\check{S}_N \coloneqq S_N \otimes_{\mathbf{C}} NZ^{-1/2}$ and $\check{S}_D \coloneqq S_D \otimes_{\mathbf{C}} NZ^{-1/2}$, and

$$S_N \coloneqq (\mathbf{R}\oplus\Lambda^2N^*Z)\otimes\Lambda T^*Z \quad \text{and} \quad S_D \coloneqq N^*Z\otimes\Lambda T^*Z.$$

The corresponding residue conditions are Lagrangian. ♠

Example 3.47. Assume that (S, γ, ∇) is a complex Dirac bundle. The **MIT bag residue conditions** are the local residue conditions arising from the decomposition

$$\check{S} = \check{S}^+ \oplus \check{S}^- \quad \text{with} \quad \check{S}^\pm \coloneqq \ker(\mathbf{1} \mp iJ);$$

cf. [Joh75]. ♠

Proposition 3.48 (Variation on the bordism theorem). *Assume the situation of Example 3.47. The components $A^\pm$ of A in the decompsition*

$$A =: \begin{pmatrix} 0 & A^- \\ A^+ & 0 \end{pmatrix}$$

satisfy

$$\operatorname{index} A^\pm = 0.$$

Proof. The following proof is essentially identical to the one presented in [BB12, §8.5]. Since $(A^+)^* = A^-$,

$$-\operatorname{index} A^- = \operatorname{index} A^+ = \dim\ker A^+ - \dim\ker A^-.$$

For every $t \in [0, 1]$, set

$$B_t^\pm \coloneqq \ker A^\pm \oplus (\mathbf{1} \mp tiJ)B_{\mathrm{APS}}.$$

Since $(B_t^+)^G = B_t^-$, by Proposition 3.42, $D_{B_t^\pm}$ is Fredholm. Moreover, by Proposition 2.28,

$$-\operatorname{index} D_{B_t^\mp} = \operatorname{index} D_{B_t^\pm} = \operatorname{index} D_{(\mathbf{1}\mp tiJ)B_{\mathrm{APS}}} + \dim\ker A^\pm.$$

In particular,

$$\operatorname{index} A^+ = 2\operatorname{index} D_{B_0^+}.$$

Therefore, it remains to prove that $\operatorname{index} D_{B_0^+} = 0$. By Proposition 2.31, it suffices to prove that $\operatorname{index} D_{B_1^+} = 0$. Since

$$B_1^\pm = H^{1/2}\Gamma(Z, \check{S}^\pm),$$

analogous to Example 2.27, for every $\phi \in \ker D_{B_1^\pm}$

$$0 = 2\langle D\phi, i\phi\rangle_{L^2} = -2\pi\langle J \operatorname{res}\phi, i\operatorname{res}\phi\rangle = \mp 2\pi\|\operatorname{res}\phi\|_{L^2}^2.$$

Therefore, $\ker D_{B_1^\pm} = \ker D_{\min}$; hence: $\operatorname{index} D_{B_1^\pm} = 0$. ■

4 Regularity theory

This section continues to assume Hypothesis 3.1 throughout. The geometric realisation $\check{H}(Z, \check{S})$ of $\check{\mathbf{H}}$ developed in Section 3 and suitable commutator estimates lead to the L^2 regularity theory laid out in the following.

4.1 Adapted Sobolev spaces, I: definition

Here is the scale of Sobolev spaces for which the regularity theory is developed.

Definition 4.1 (Differential operators). Denote by $\mathrm{DiffOp}^\bullet(S \otimes \mathfrak{I})$ the $\mathbf{N}_0$–filtered ring of differential operators acting on $S \otimes \mathfrak{I}$.

1. A vector field $v \in \mathrm{Vect}(\hat{X})$ is **conormal** if $v|_{\partial\hat{X}} \in \mathrm{Vect}(\partial\hat{X})$. Denote the subspace of conormal vector fields by $\mathrm{Vect}_b(X\backslash Z)$.
2. The filtered subring $\mathrm{DiffOp}_b^\bullet(S \otimes \mathfrak{I}) \subset \mathrm{DiffOp}^\bullet(S \otimes \mathfrak{I})$ of **conormal** differential operators is generated by $\Gamma(\hat{X}, \mathrm{End}(\hat{S} \otimes \hat{\mathfrak{I}}))$ and differential operators of the form ∇_v with $v \in \mathrm{Vect}_b(\hat{X})$.
3. The filtered subring $\mathrm{DiffOp}_a^\bullet(S \otimes \mathfrak{I}) \subset \mathrm{DiffOp}^\bullet(S \otimes \mathfrak{I})$ of **adapted** differential operators is generated by $\mathrm{DiffOp}_b^\bullet(S \otimes \mathfrak{I})$ and D. ●

Definition 4.2 (Sobolev spaces). Let $k \in \mathbf{N}_0$.

(b) The **conormal Sobolev space** $H_b^k\Gamma(X\backslash Z, S \otimes \mathfrak{I})$ is defined by

$$H_b^k\Gamma(X\backslash Z, S \otimes \mathfrak{I}) \coloneqq \left\{ \phi \in H_{\mathrm{loc}}^k\Gamma(X\backslash Z, S \otimes \mathfrak{I}) : \begin{array}{r} P\phi \in L^2\Gamma(X\backslash Z, S \otimes \mathfrak{I}) \text{ for} \\ \text{every } P \in \mathrm{DiffOp}_b^k(S \otimes \mathfrak{I}) \end{array} \right\}.$$

Choose a finite subset $\mathscr{P}_b^k \subset \mathrm{DiffOp}_b^k(S \otimes \mathfrak{I})$ which spans $\mathrm{DiffOp}_b^k(S \otimes \mathfrak{I})$ over $\Gamma(\hat{X}, \mathrm{End}(\hat{S} \otimes \hat{\mathfrak{I}}))$. Define the norm $\|-\|_{H_b^k} : H_b^k\Gamma(X\backslash Z, S \otimes \mathfrak{I}) \to [0, \infty)$ by

$$\|\phi\|_{H_b^k}^2 \coloneqq \sum_{P \in \mathscr{P}_b^k} \|P\phi\|_{L^2}^2.$$

(a) The **adapted Sobolev space** $H_a^k\Gamma(X\backslash Z, S \otimes \mathfrak{I})$ is defined by

$$H_a^k\Gamma(X\backslash Z, S \otimes \mathfrak{I}) \coloneqq \left\{ \phi \in H_{\mathrm{loc}}^k\Gamma(X\backslash Z, S \otimes \mathfrak{I}) : \begin{array}{r} P\phi \in L^2\Gamma(X\backslash Z, S \otimes \mathfrak{I}) \text{ for} \\ \text{every } P \in \mathrm{DiffOp}_a^k(S \otimes \mathfrak{I}) \end{array} \right\}.$$

Choose a finite subset $\mathscr{P}_a^k \subset \mathrm{DiffOp}_a^k(S \otimes \mathfrak{I})$ which spans $\mathrm{DiffOp}_a^k(S \otimes \mathfrak{I})$ over $\Gamma(\hat{X}, \mathrm{End}(\hat{S} \otimes \hat{\mathfrak{I}}))$. Define the norm $\|-\|_{H_a^k} : H_a^k\Gamma(X\backslash Z, S \otimes \mathfrak{I}) \to [0, \infty)$ by

$$\|\phi\|_{H_a^k}^2 \coloneqq \sum_{P \in \mathscr{P}_a^k} \|P\phi\|_{L^2}^2. \qquad \bullet$$

$(H_b^k\Gamma(X\backslash Z, S \otimes \mathfrak{I}), \|-\|_{H_b^k})$ and $(H_a^k\Gamma(X\backslash Z, S \otimes \mathfrak{I}), \|-\|_{H_a^k})$ are Hilbert spaces. Evidently, different choices of $\mathscr{P}_b^k$, $\mathscr{P}_a^k$ lead to equivalent norms. The following discussion leads to particularly convenient choices of $\mathscr{P}_b^k$, $\mathscr{P}_a^k$.

Definition 4.3 (Convenient vector fields).

1. Denote by $\mathrm{Vect}_c(X\backslash Z) \subset \mathrm{Vect}_b(\hat{X})$ the subspace of vector fields supported in $X\backslash Z \subset \hat{X}$.

(2) Denote by $\mathrm{Vect}_{b;c}(\hat{U}) \subset \mathrm{Vect}_b(\hat{X})$ the subspace of vector fields supported in $\hat{U} \subset \hat{X}$. For $v \in \mathrm{Vect}_{b;c}(\hat{U})$, $\mathring{\nabla}_v \in \mathrm{DiffOp}_b^1(S \otimes \mathfrak{l})$.

(3) Denote by $\mathrm{Vect}_{b;c,0}(\hat{U})$ the subspace of those $v \in \mathrm{Vect}_{b;c}(\hat{U})$ which are $\mathrm{U}(1)$–invariant on ∂X; that is: $[\partial_\alpha, v|_{\partial\hat{X}}] = 0$. •

Remark 4.4. $\mathrm{Vect}_{b;c}(\hat{U})$ is generated by $\chi(r)\partial_\alpha$, $\chi(r)r\partial_r$, vector field of the form $\chi(r)v$ where v is lifted from Z, and vector fields vanishing near Z. ♣

Lemma 4.5 (Commutation relations). *The following commutation relations hold:*

(1) *For every* $v \in \mathrm{Vect}_c(X\backslash Z)$*,* $k \in \mathbf{N}$*, and* $P \in \mathrm{DiffOp}_a^k(S \otimes \mathfrak{l})$

$$[\nabla_v, P] \in \mathrm{DiffOp}_b^k(S \otimes \mathfrak{l}).$$

(2) *For every* $v \in \mathrm{Vect}_{b;c}(\hat{U})$*,* $k \in \mathbf{N}$*, and* $P \in \mathrm{DiffOp}_b^k(S \otimes \mathfrak{l})$

$$[\mathring{\nabla}_v, P] \in \mathrm{DiffOp}_b^k(S \otimes \mathfrak{l}).$$

(3) *For every* $v \in \mathrm{Vect}_{b;c,0}(\hat{U})$

$$[\mathring{\nabla}_v, D] \in \mathrm{DiffOp}_b^1(S \otimes \mathfrak{l}) + \mathrm{DiffOp}_b^0(S \otimes \mathfrak{l}) \cdot D.$$

Proof. If $v, w \in \mathrm{Vect}(X\backslash Z)$ and $T \in \Gamma(X\backslash Z, S \otimes \mathfrak{l})$, then

$$[\nabla_v, T] = \nabla_v T \quad \text{and} \quad [\nabla_v, \nabla_w] = \nabla_{[v,w]} + F_\nabla(v, w).$$

Therefore, for every $k \in \mathbf{N}$ and $P \in \mathrm{DiffOp}^k(S \otimes \mathfrak{l})$, $[\nabla_v, P] \in \mathrm{DiffOp}^k(S \otimes \mathfrak{l})$.

If $v \in \mathrm{Vect}_c(X\backslash Z)$, $k \in \mathbf{N}$, and $P \in \mathrm{DiffOp}_a^k(S \otimes \mathfrak{l})$, then $\mathrm{supp}([\nabla_v, P]) \subset X\backslash Z$; therefore and by the above observation, $[\nabla_v, P] \in \mathrm{DiffOp}_b^k(S \otimes \mathfrak{l})$. This proves (1).

(2) is immediate from the above observation.

Let $v \in \mathrm{Vect}_{b;c,0}(\hat{U})$. By Proposition 3.14, $D - \chi(r) \cdot \mathring{D} \in \mathrm{DiffOp}_b^1(S \otimes \mathfrak{l})$. Therefore, it suffices to prove that

$$[\mathring{\nabla}_v, \chi(r) \cdot \mathring{D}] \in \mathrm{DiffOp}_b^1(S \otimes \mathfrak{l}) + \mathrm{DiffOp}_b^0(S \otimes \mathfrak{l}) \cdot \chi(r) \cdot \mathring{D}.$$

By direct computation,

$$[\mathring{\nabla}_{r\partial_r}, \mathring{\nabla}_{\partial_r} - r^{-1}I\mathring{\nabla}_{\partial_\alpha}] = -(\mathring{\nabla}_{\partial_r} - r^{-1}I\mathring{\nabla}_{\partial_\alpha}) \quad \text{and} \quad [\mathring{\nabla}_{\partial_\alpha}, \mathring{\nabla}_{\partial_r} - r^{-1}I\mathring{\nabla}_{\partial_\alpha}] = 0;$$

moreover, if v is the lift of a vector field along Z, then

$$[\mathring{\nabla}_v, \mathring{\nabla}_{\partial_r} - r^{-1}I\mathring{\nabla}_{\partial_\alpha}] = 0.$$

By Remark 3.13, Remark 4.4 and since $\chi(r) \cdot D_Z \in \mathrm{DiffOp}_b^1(S \otimes \mathfrak{l})$ and $\chi(r) \cdot J \in \mathrm{DiffOp}_b^0(S \otimes \mathfrak{l})$, this implies (3). ■

Corollary 4.6 (Convenient choices of $\mathscr{P}_b^k$, $\mathscr{P}_a^k$).

(1) *Set* $\mathscr{P}_b^0 \coloneqq \{\mathrm{id}_{S\otimes\mathfrak{I}}\}$. *For every* $k \in \mathbf{N}$ *there is a finite subset* $\mathscr{P}_b^k \subset \mathrm{DiffOp}_b^k(S\otimes\mathfrak{I})$ *which spans* $\mathrm{DiffOp}_b^k(S\otimes\mathfrak{I})$ *over* $\Gamma(\hat{X}, \mathrm{End}(\hat{S}\otimes\hat{\mathfrak{I}}))$ *such that* $\mathscr{P}_b^{k-1} \subset \mathscr{P}_b^k$ *and every* $P \in \mathscr{P}_b^k \backslash \mathscr{P}_b^{k-1}$ *is of the form*

$$P = \nabla_{u_1} \cdots \nabla_{u_\ell} \mathring{\nabla}_{v_1} \cdots \mathring{\nabla}_{v_{k-\ell}}$$

with

$$u_1, \dots, u_\ell \in \mathrm{Vect}_c(X\backslash Z) \quad \textit{and} \quad v_1, \dots, v_{k-\ell} \in \mathrm{Vect}_{b;c,0}(\hat{U}).$$

(2) *Let* $k \in \mathbf{N}$. *If* $\mathscr{P}_b^1, \dots, \mathscr{P}_b^k$ *are as above then*

$$\mathscr{P}_a^k \coloneqq \bigcup_{\ell=0}^{k} \{PD^\ell : P \in \mathscr{P}_b^{k-\ell}\}$$

spans $\mathrm{DiffOp}_a^k(S\otimes\mathfrak{I})$ *over* $\Gamma(\hat{X}, \mathrm{End}(\hat{S}\otimes\hat{\mathfrak{I}}))$. ■

Henceforth, for every $k \in \mathbf{N}$, $\mathscr{P}_a^k$, $\mathscr{P}_b^k$ are assumed to be chosen as in Corollary 4.6; in particular,

$$\|\phi\|_{H_a^k}^2 = \sum_{\ell=0}^{k} \|D^\ell\phi\|_{H_b^{k-\ell}}^2. \tag{4.7}$$

Remark 4.8. Let (W, g) be a Riemannian manifold with boundary equipped with a Dirac bundle (S, γ, ∇). The conormal Sobolev space $H_b^k\Gamma(W, S)$; see, e.g., [Mel93, (5.42)]; is an important tool in the study of boundary values problems for Dirac operators. A moment's thought with the above discussion in mind shows that $H_a^k\Gamma(W, S)$, the analogue of the adapted Sobolev space, agrees with the usual Sobolev space $H^k\Gamma(W, S)$. Indeed, these are the appropriate Sobolev spaces for the L^2 regularity theory; see [BB12, §6.2]. ♣

Remark 4.9. Let $k \in \mathbf{N}_0$ and $\phi \in (L^2 \cap H_{\mathrm{loc}}^k)\Gamma(X\backslash Z, S\otimes\mathfrak{I})$ with $\mathrm{supp}(\phi) \subset U\backslash Z$. Decompose

$$L^2\Gamma(U\backslash Z, \mathring{S}\otimes\mathring{\mathfrak{I}}) \ni \phi = \sum_{(\lambda,\mu)\in\check{\sigma}} \phi_{\lambda,\mu} \in \bigoplus_{(\lambda,\mu)\in\sigma} L^2((0,1), r\mathrm{d}r; E_{\lambda,\mu})$$

as in Section 3.4. By direct inspection, $\phi \in H_b^k\Gamma(X\backslash Z, S\otimes\mathfrak{I})$ if and only if

$$\sum_{\ell=0}^{k} \sum_{(\lambda,\mu)\in\sigma} \int_0^1 (\langle\lambda\rangle + \langle\mu\rangle)^{2(k-\ell)} |(r\partial_r)^\ell \phi_{\lambda,\mu}(r)|^2 \, r\mathrm{d}r < \infty;$$

indeed, for uniformly equivalent to $\|\phi\|_{H_b^k}^2$. The crucial point is v is a vector field lifted from Z to F, then for every $\phi \in V_{\lambda,\mu}$

$$\|\nabla_v\phi\| \lesssim_v \langle\mu\rangle\|\phi\|.$$

Proposition 3.25 yields an analogous characterisation of $\phi \in H_a^k\Gamma(X\backslash Z, S\otimes\mathfrak{I})$ and description of $\|\phi\|_{H_a^k}$. ♣

4.2 Elliptic regularity and estimates

Here is the fundamental regularity result.

Theorem 4.10 (Elliptic regularity and estimates, I). *For every $k \in \mathbf{N}_0$*

$$H_a^{k+1}\Gamma(X\backslash Z, S \otimes \mathfrak{l}) = \left\{ \phi \in H_{\mathrm{loc}}^{k+1}\Gamma(X\backslash Z, S \otimes \mathfrak{l}) : \begin{matrix} \phi, D\phi \in H_a^k\Gamma(X\backslash Z, S \otimes \mathfrak{l}) \\ \text{and } \operatorname{res}[\phi] \in H^{k+\frac{1}{2}}\Gamma(Z, \check{S}) \end{matrix} \right\}$$

moreover: for every $\phi \in H_a^{k+1}\Gamma(X\backslash Z, S \otimes \mathfrak{l})$

$$\|\phi\|_{H_a^{k+1}} \asymp_k \|D\phi\|_{H_a^k} + \|\phi\|_{H_a^k} + \|\operatorname{res}[\phi]\|_{H^{k+1/2}}.$$

The proof relies on the following observations.

Lemma 4.11. *For every $k \in \mathbf{N}_0$ the extension map* $\operatorname{ext}\colon \check{H}\Gamma(Z, \check{S}) \to \operatorname{dom}(D_{\max})$ *restricts to a bounded injective linear map with closed image:*

$$\operatorname{ext}\colon H^{k+1/2}\Gamma(Z, \check{S}) \to H_a^{k+1}\Gamma(X\backslash Z, S \otimes \mathfrak{l}).$$

Proof. By (4.7), Proposition 3.14 and Lemma 4.5, it suffices to prove that for every $(-1/2, \mu) \in \check{\sigma}$ and $\phi \in V_{-1/2,\mu} \subset \check{H}\Gamma(Z, \check{S})$

$$\sum_{\ell=0}^{k+1} \|\mathring{D}^\ell \operatorname{ext}\phi\|_{H_b^{k-\ell+1}}^2 = \sum_{\ell=0}^{k+1} \|\mathring{D}^\ell(\chi(r)r^{-1/2}e^{-|\mu|r}\phi)\|_{H_b^{k-\ell+1}}^2 \asymp_k \langle\mu\rangle^{2k+1}\|\phi\|^2 \asymp_k \|\phi\|_{H^{k+1/2}}^2.$$

Let $f \in C^\infty([0,1), \mathbf{R})$. By direct computation,

$$\begin{aligned} \mathring{\nabla}_{r\partial_r}(f(r)r^{-1/2}e^{-|\mu|r}\phi) &= \big(rf'(r) - (\tfrac{1}{2} + |\mu|r)f(r)\big)r^{-1/2}e^{-|\mu|r}\phi \quad \text{and} \\ \mathring{\nabla}_{\partial_\alpha}(f(r)r^{-1/2}e^{-|\mu|r}\phi) &= \tfrac{1}{2}f(r)r^{-1/2}e^{-|\mu|r}I\phi. \end{aligned}$$

Therefore and by Definition 3.32,

$$\begin{aligned} \mathring{D}(f(r)r^{-1/2}e^{-|\mu|r}\phi) &= J(\partial_r + \tfrac{1}{2r} + A_\mu)(f(r)r^{-1/2}e^{-|\mu|r}\phi) \\ &= \big(f'(r) - (|\mu| + A_\mu)f(r)\big)r^{-1/2}e^{-|\mu|r}J\phi. \end{aligned}$$

Consequently, for every $\ell, m, n \in \mathbf{N}_0$

$$(\mathring{\nabla}_{\partial_\alpha})^n(\mathring{\nabla}_{r\partial_r})^m\mathring{D}^\ell\big(\chi(r)r^{-1/2}e^{-|\mu|r}\phi\big) = 2^{-n}f_{m,\ell}(r)r^{-1/2}e^{-|\mu|r}I^nJ^\ell\phi.$$

where $f_{m,\ell} \in C^\infty([0,\infty), \mathbf{R})$ are recursively defined by

$$f_{m,\ell}(r) \coloneqq \begin{cases} \chi(r) & \text{if } m = \ell = 0 \\ f'_{m,\ell-1}(r) - (|\mu| + A_\mu)f_{m,\ell-1} & \text{if } m = 0 \text{ and } \ell \geqslant 1 \\ rf'_{m-1,\ell}(r) - (\tfrac{1}{2} + |\mu|r)f_{m-1,\ell}(r) & \text{if } m \geqslant 1. \end{cases}$$

A brief computation shows that

$$\int_0^1 f_{m,\ell}^2(r)e^{-2|\mu|r}\mathrm{d}r \lesssim_{m,\ell} \langle\mu\rangle^{2(m+\ell)-1}.$$

Therefore, if $v_1, \dots, v_o$ are lifts of vector fields along Z, then

$$\begin{aligned}\|\mathring{\nabla}_{v_1}\dots\mathring{\nabla}_{v_o}(\mathring{\nabla}_{\partial_\alpha})^n(\mathring{\nabla}_{r\partial_r})^m\mathring{D}^\ell \operatorname{ext}\phi\|_{L^2}^2 &\lesssim_o \int_0^1 f_{m,\ell}^2(r)e^{-2|\mu|r}\mathrm{d}r \cdot (\|\phi\|^2 + \|A^o\phi\|^2) \\ &\asymp_o \langle\mu\rangle^{2o}\int_0^1 f_{m,\ell}^2(r)e^{-2|\mu|r}\mathrm{d}r \cdot \|\phi\|^2.\end{aligned}$$

In light of Corollary 4.6 this proves the assertion. ■

Lemma 4.12. *For every* $k \in \mathbf{N}_0$ *the residue map* $\operatorname{res}\colon \operatorname{dom}(D_{\max}) \to \check{H}\Gamma(Z,\check{S})$ *restricts to a bounded surjective linear map*

$$\operatorname{res}\colon H_a^{k+1}\Gamma(X\backslash Z, S\otimes\mathfrak{l}) \to H^{k+1/2}\Gamma(Z,\check{S}).$$

Proof. Since $H^1\Gamma(X\backslash Z, S\otimes\mathfrak{l}) \hookrightarrow H_a^1\Gamma(X\backslash Z, S\otimes\mathfrak{l})$ with closed image by (2.6), and by Lemma 4.11,

$$H_a^1\Gamma(X\backslash Z, S\otimes\mathfrak{l}) = H^1\Gamma(X\backslash Z, S\otimes\mathfrak{l}) \oplus \operatorname{ext}(H^{1/2}\Gamma(Z,\check{S})).$$

This proves the assertion for $k = 0$, again by Lemma 4.11.

If $\phi \in H_a^{k+1}\Gamma(X\backslash Z, S\otimes\mathfrak{l})$, then $\chi(r)A^k\phi \in H_a^1\Gamma(X\backslash Z, S\otimes\mathfrak{l})$ because $\chi(r)A^k \in \operatorname{DiffOp}_b^k(S\otimes\mathfrak{l})$. Evidently,

$$A^k \operatorname{res}[\phi] = \operatorname{res}[\chi(r)A^k\phi];$$

cf. Remark 4.9. Therefore, the assertion holds for every $k \in \mathbf{N}_0$. ■

Lemma 4.13. *Let* $k \in \mathbf{N}$. *Let* $\phi \in H_{\mathrm{loc}}^{k+1}\Gamma(X\backslash Z, S\otimes\mathfrak{l})$. *If* $\phi, D\phi \in H_a^k\Gamma(X\backslash Z, S\otimes\mathfrak{l})$ *and* $\operatorname{res}[\phi] = 0$, *then, for every* $P \in \mathscr{P}_b^k$, $P\phi \in H^1\Gamma(X\backslash Z, S\otimes\mathfrak{l})$.

Proof. By Lemma 4.5, $P\phi \in \operatorname{dom}(D_{\max})$. Therefore, it remains to prove that $\operatorname{res}[P\phi] = 0$. In fact, by induction, it suffices to prove this for $k = 1$.

If $P = \chi(r)\cdot\mathring{\nabla}_{\partial_\alpha}$ or $P = \chi(r)\cdot\mathring{\nabla}_v$ as in Remark 4.4, then this is evident from Remark 4.9. It remains to consider $P = \chi(r)\cdot r\mathring{\nabla}_{\partial_r}$ or, in fact, $P = \chi(r)\cdot r\mathring{D}$. Since $Q \coloneqq \chi(r)\mathring{D} - D \in \operatorname{DiffOp}_b^1(S\otimes\mathfrak{l})$,

$$D\phi + Q\phi \in \operatorname{dom}(D_{\max}).$$

Therefore, $P\phi = r(D\phi + Q\phi) \in H^1\Gamma(X\backslash Z, S\otimes\mathfrak{l})$. ■

Proof of Theorem 4.10. Let $k \in \mathbf{N}_0$. By Lemma 4.12, it suffices to prove that for every $\phi \in H_{\mathrm{loc}}^{k+1}\Gamma(X\backslash Z, S\otimes\mathfrak{l})$ with $\phi, D\phi \in H_a^k\Gamma(X\backslash Z, S\otimes\mathfrak{l})$ and $\operatorname{res}[\phi] \in H^{k+1/2}\Gamma(Z,\check{S})$

$$\|\phi\|_{H_a^{k+1}} \lesssim_k \|D\phi\|_{H_a^k} + \|\phi\|_{H_a^k} + \|\operatorname{res}[\phi]\|_{H^{k+1/2}}.$$

Since

$$\phi = (\phi - \operatorname{ext}\operatorname{res}[\phi]) + \operatorname{ext}\operatorname{res}[\phi]$$

and by Lemma 4.11, it suffices to prove that above assuming $\operatorname{res}[\phi] = 0$.

Since $H^1\Gamma(X\backslash Z, S\otimes \mathfrak{l}) \hookrightarrow H^1_a\Gamma(X\backslash Z, S\otimes \mathfrak{l})$ and by (2.6), the assertion holds for $k = 0$.

Suppose that $k \in \mathbf{N}$. By Lemma 4.13 and Lemma 4.5, for every $P \in \mathscr{P}^k_b$

$$\|P\phi\|_{H^1_a} \lesssim \|DP\phi\|_{L^2} + \|P\phi\|_{L^2} \lesssim_P \|D\phi\|_{H^k_a} + \|\phi\|_{H^k_a}.$$

This implies the assertion. ■

For suitable residue conditions $B \subset \check{H}\Gamma(Z, \check{S})$, the term $\operatorname{res}[\phi]$ in Theorem 4.10 can be absorbed provided $\operatorname{res}[\phi] \in B$.

Definition 4.14. Let $B \subset \check{H}\Gamma(Z, \check{S})$ be a residue condition.

(1) Let $k \in \mathbf{N}_0$. B is $(k + \frac{1}{2})$**–regular** if for every $\phi \in B$

$$\|\phi\|_{H^{k+1/2}} \lesssim_{B,k} \|\mathbf{1}_{(-\infty,0)}(A)\phi\|_{H^{k+1/2}} + \|\phi\|_{\check{H}}.$$

(2) B is ∞**–regular** if it is $(k + 1/2)$–regular for every $k \in \mathbf{N}_0$. •

Example 4.15. The APS residue condition B_{APS} is ∞–regular. ♠

Theorem 4.16 (Elliptic regularity and estimates, II). *Let $k \in \mathbf{N}_0$. Let B be a $(k+\frac{1}{2})$–regular residue condition. If $\phi \in H^{k+1}_{\mathrm{loc}}\Gamma(X\backslash Z, S\otimes \mathfrak{l})$ satisfies $\phi, D\phi \in H^k_a\Gamma(X\backslash Z, S\otimes \mathfrak{l})$ and* $\operatorname{res}[\phi] \in B$*, then $\phi \in H^{k+1}_a\Gamma(X\backslash Z, S\otimes \mathfrak{l})$ and*

$$\|\phi\|_{H^{k+1}_a} \asymp_{B,k} \|D\phi\|_{H^k_a} + \|\phi\|_{L^2}.$$

The proof requires the following preparation.

Lemma 4.17. *For every $k \in \mathbf{N}_0$ and $\phi \in$* $\operatorname{dom}(D_{\max})$

$$\|\mathbf{1}_{(-\infty,0)}(A)\operatorname{res}[\phi]\|_{H^{k+1/2}} \lesssim_k \|D\phi\|_{H^k_b} + \|\phi\|_{H^k_b}.$$

Proof. Since $\operatorname{res}\circ[\cdot] \colon \operatorname{dom}(D_{\max}) \to \check{H}\Gamma(Z, \check{S})$ is bounded and $\chi(r)A^k \in \operatorname{DiffOp}^k_b(S\otimes \mathfrak{l})$,

$$\begin{aligned}\|\mathbf{1}_{(-\infty,0)}(A)\operatorname{res}[\phi]\|_{H^{k+1/2}} &\lesssim_k \|A^k\operatorname{res}[\phi]\|_{\check{H}} = \|\operatorname{res}[\chi(r)\cdot A^k\phi]\|_{\check{H}} \\ &\lesssim \|\chi(r)\cdot A^k\phi\|_D \lesssim_k \|D\phi\|_{H^k_b} + \|\phi\|_{H^k_b}\end{aligned}$$

by Corollary 4.6. ■

Proof of Theorem 4.16. By Lemma 4.17 and since B is $(k + \frac{1}{2})$–regular, for every $\phi \in \operatorname{dom}(D_B)$

$$\|\operatorname{res}[\phi]\|_{H^{k+1/2}} \lesssim_{B,k} \|\mathbf{1}_{(-\infty,0)}(A)\operatorname{res}[\phi]\|_{H^{k+1/2}} + \|\operatorname{res}[\phi]\|_{\check{H}} \lesssim_k \|D\phi\|_{H^k_b} + \|\phi\|_{H^k_b}.$$

This together with Theorem 4.16 implies the assertion. ■

4.3 Fredholm extensions in higher regularity

The following is a consequence of Proposition 3.42.

Corollary 4.18 ($\frac{1}{2}$–regular implies left semi-Fredholm). *Let $B \subset \check{H}\Gamma(Z, \check{S})$ be a residue condition. If B is $\frac{1}{2}$–regular, then D_B is left semi-Fredholm.* ■

The discussion in Section 4.2 leads to the following observation.

Definition 4.19. Let $k \in \mathbf{N}_0$. Let $B \subset \check{H}\Gamma(Z, \check{S})$ be a residue condition. Consider the closed subspace

$$H_a^{k+1}\Gamma(X \backslash Z, S \otimes \mathfrak{l}; B) \coloneqq \{\phi \in H_a^{k+1}\Gamma(X \backslash Z, S \otimes \mathfrak{l}) : \operatorname{res}[\phi] \in B\}$$

and the restriction of D to

$$D_{B,k} \colon\ H_a^{k+1}\Gamma(X \backslash Z, S \otimes \mathfrak{l}; B) \to H_a^{k}\Gamma(X \backslash Z, S \otimes \mathfrak{l}).$$ •

Proposition 4.20. *Let $k \in \mathbf{N}_0$. Let $B \subset \check{H}\Gamma(Z, \check{S})$ is a $(k + \frac{1}{2})$–regular residue condition, then $D_{B,k}$ is left semi-Fredholm; in fact:*

$$\ker D_{B,k} = \ker D_B$$

and the canonical map

$$\operatorname{coker} D_{B,k} \to \operatorname{coker} D_B \cong (\ker D_{B^G})^*$$

is an isomorphism; moreover, if $\ker D_{B^G} \subset H_a^k\Gamma(X \backslash Z, S \otimes \mathfrak{l})$, then the latter L^2 orthogonally decomposes as

$$H_a^k\Gamma(X \backslash Z, S \otimes \mathfrak{l}) = \operatorname{im} D_{B,k} \oplus \ker D_{B^G}.$$

Proof. The proof is identical to the one of [DW24, Theorem 3.57], but repeated here for the readers' convenience. By Theorem 4.16, $\ker D_{B,k} = \ker D_B$; moreover: the linear map

$$\frac{\operatorname{dom}(D_B)}{H_a^{k+1}\Gamma(X \backslash Z, S \otimes \mathfrak{l}; B)} \to \frac{L^2\Gamma(X \backslash Z, S \otimes \mathfrak{l})}{H_a^{k}\Gamma(X \backslash Z, S \otimes \mathfrak{l})}$$

induced by D_B is injective. Therefore, by the Snake Lemma, the canonical map

$$\operatorname{coker} D_{B,k} \to \operatorname{coker} D_B$$

is injective.

Since $H_a^k\Gamma(X \backslash Z, S \otimes \mathfrak{l})$ is dense in $L^2\Gamma(X \backslash Z, S \otimes \mathfrak{l})$ the map

$$H_a^k\Gamma(X \backslash Z, S \otimes \mathfrak{l}) \to (\ker D_{B^G})^* \cong \operatorname{coker} D_B$$

is surjective. Since it factors through $\operatorname{coker} D_{B,k} \to \operatorname{coker} D_B$, the latter must be surjective. ■

Proposition 4.21. *For every $k \in \mathbf{N}_0$ the restriction of D to*

$$D_k \colon\ H_a^{k+1}\Gamma(X \backslash Z, S \otimes \mathfrak{l}) \to H_a^{k}\Gamma(X \backslash Z, S \otimes \mathfrak{l})$$

is right semi-Fredholm; that is: $\operatorname{im} D_k$ is closed and $\operatorname{coker} D_k$ is finite-dimensional; moreover: $H_a^k\Gamma(X \backslash Z, S \otimes \mathfrak{l})$ L^2 orthogonally decomposes as

$$H_a^k\Gamma(X \backslash Z, S \otimes \mathfrak{l}) = \operatorname{im} D_k \oplus \ker D_{\min}.$$

Proof. Since $\ker D_{B_{\mathrm{APS}}}$ is finite-dimensional, there is a $\tau \leqslant 0$ such that the projection

$$\operatorname{res}(\ker D_{B_{\mathrm{APS}}}) \to \bigoplus_{\lambda \in [\tau, 0)} \ker(A - \lambda \mathbf{1})$$

is injective. The residue conditions

$$B_\tau \coloneqq \mathbf{1}_{(-\infty,\tau)}(A) H^{1/2}\Gamma(Z, \check{S}) \subset \check{H}\Gamma(Z, \check{S})$$

and B_τ^G are ∞–regular. Therefore, by Proposition 4.20,

$$H_a^k\Gamma(X \backslash Z, S \otimes \mathfrak{I}) = \operatorname{im} D_{B_\tau^G, k} \oplus \ker D_{B_\tau, k}.$$

By construction, $\ker D_{B_\tau, k} = \ker D_{\min}$. Moreover, since $\operatorname{im} D_k \perp \ker D_{\min}$, $\operatorname{im} D_{B_\tau^G, k} = \operatorname{im} D_k$. ■

4.4 Adapted Sobolev spaces, II: Morrey embedding and polyhomogeneity

The purpose of the upcoming two subsections is to further understand the scale of adapted Sobolev spaces $(H_a^k\Gamma(X \backslash Z, S \otimes \mathfrak{I}), \|-\|_{H_a^k})_{k \in \mathbf{N}_0}$. A crucial observation is that the singularities in the elements of $H_a^k\Gamma(X \backslash Z, S \otimes \mathfrak{I})$ can be removed after untwisting in the following sense.

Definition 4.22. Define the **twist** $\bar{z}^{-1/2} \in \Gamma(U \backslash Z, \mathrm{Hom}_{\mathbf{C}}(\Pi^*\check{S}, \mathring{S} \otimes \mathring{\mathfrak{I}}))$ by

$$\bar{z}^{-1/2}\phi \coloneqq r^{-1/2}P_{-1/2}\phi$$

with $P_{-1/2}$ as in Proposition 3.21. •

Lemma 4.23 (Removable singularities after untwisting). *For every $k \in \mathbf{N}_0$ the restriction map $H_a^k\Gamma(X \backslash Z, S \otimes \mathfrak{I}) \to H_{\mathrm{loc}}^k\Gamma(U \backslash Z, \mathring{S} \otimes \mathring{\mathfrak{I}})$ factors through*

$$\bar{z}^{-1/2}H^k\Gamma(U, \Pi^*\check{S}) \subset H_{\mathrm{loc}}^k\Gamma(U \backslash Z, \mathring{S} \otimes \mathring{\mathfrak{I}}).$$

Proof. Let $k \in \mathbf{N}_0$ and $\phi \in H_a^k\Gamma(X \backslash Z, S \otimes \mathfrak{I})$. Set

$$\psi \coloneqq \bar{z}^{1/2}\phi \in H_{\mathrm{loc}}^k\Gamma(U \backslash Z, \Pi^*\check{S}).$$

Since $|\bar{z}^{1/2}|$ is bounded, $\psi \in r^{1/2}L^2\Gamma(U, \Pi^*\check{S})$. This proves the assertion for $k = 0$.

Henceforth, suppose that $k \in \mathbf{N}$. Consider the differential operators $\mathfrak{d}_z \colon H_{\mathrm{loc}}^1\Gamma(U, \Pi^*\check{S}) \to L^2\Gamma(U, \Pi^*\check{S})$ and $\mathring{\mathfrak{d}}_z \colon H_{\mathrm{loc}}^1\Gamma(U \backslash Z, \mathring{S} \otimes \mathring{\mathfrak{I}}) \to L^2\Gamma(U \backslash Z, \mathring{S} \otimes \mathring{\mathfrak{I}})$ defined by

$$\mathfrak{d}_z \coloneqq J(\partial_r - r^{-1}I\nabla_{\partial_\alpha}) \quad \text{and} \quad \mathring{\mathfrak{d}}_z \coloneqq J(\partial_r - r^{-1}I\mathring{\nabla}_{\partial_\alpha}).$$

The difference $\mathring{D} - \mathring{\mathfrak{d}}_z$ is a first order conormal differential operator; see the proof of Lemma 4.5. If v is the lift of a vector field on Z, then

$$\nabla_v\psi = \bar{z}^{1/2}\mathring{\nabla}_v\phi \quad \text{and} \quad \mathfrak{d}_z\psi = \bar{z}^{1/2}\mathring{\mathfrak{d}}_z\phi$$

on $U \backslash Z$.

A moment's thought shows that $\nabla_v\psi = \bar{z}^{1/2}\mathring{\nabla}_v\phi$ holds on U in the sense of distributions. In fact, $\mathfrak{d}_z\psi = \bar{z}^{1/2}\mathring{\mathfrak{d}}_z\phi$ also holds on U in the sense of distributions. To see this let η_ε be a suitable cut-off function and τ a test section. By direct computation,

$$\int_U \langle \eta_\varepsilon\tau, \mathfrak{d}_z\psi\rangle = \int_U \langle \eta_\varepsilon\mathfrak{d}_z^*\tau, \psi\rangle + \langle \sigma_{\mathfrak{d}_z}(\mathrm{d}\eta_\varepsilon)\tau, \psi\rangle$$

and

$$\left|\int_U \langle \sigma_{\mathfrak{d}_z}(\mathrm{d}\eta_\varepsilon)\tau, \psi\rangle\right| \lesssim_\tau \int_U r^{1/2}|\mathrm{d}\eta_\varepsilon| r^{-1/2}|\psi| \lesssim_\psi \int_U r|\mathrm{d}\eta_\varepsilon|^2.$$

Since η_ε can be chosen so that $\mathrm{d}\eta_\varepsilon$ is supported in $B_{2\varepsilon}(Z)$ and $r|\mathrm{d}\eta_\varepsilon| \lesssim 1$, it follows that

$$\int_U \langle \tau, \mathfrak{d}_z\psi\rangle - \langle \mathfrak{d}_z^*\tau, \psi\rangle = \lim_{\varepsilon\downarrow 0}\int_U \eta_\varepsilon(\langle \tau, \mathfrak{d}_z\psi\rangle - \langle \mathfrak{d}_z^*\tau, \psi\rangle) = 0.$$

By induction, it follows that if $v_1, \ldots, v_{k-\ell}$ are lifts of vector fields on Z, then

$$\nabla_{v_1}\ldots\nabla_{v_{k-\ell}}\mathfrak{d}_z^\ell\psi = \bar{z}^{1/2}\mathring{\nabla}_{v_1}\ldots\mathring{\nabla}_{v_{k-\ell}}\mathring{\mathfrak{d}}_z^\ell\phi$$

holds on U in the sense of distributions; in particular:

$$\nabla_{v_1}\ldots\nabla_{v_{k-\ell}}\mathfrak{d}_z^\ell\psi \in r^{1/2}L^2\Gamma(U, \Pi^*\check{S}).$$

This implies that $\psi \in H^k\Gamma(U, \Pi^*\check{S})$ because

$$\int_U |\mathfrak{d}_z(\chi\psi)|^2 = \int_U |\partial_r(\chi\psi)|^2 + r^{-2}|\nabla_{\partial_\alpha}(\chi\psi)|^2 + O(|\psi|^2). \qquad \blacksquare$$

Set

$$H_a^\infty\Gamma(X\backslash Z, S\otimes\mathfrak{l}) \coloneqq \bigcap_{k\in\mathbf{N}_0} H_a^k\Gamma(X\backslash Z, S\otimes\mathfrak{l}).$$

Corollary 4.24 (Smooth after untwisting). *The restriction map* $H_a^\infty\Gamma(X\backslash Z, S\otimes\mathfrak{l}) \to \Gamma(U\backslash Z, \mathring{S}\otimes\mathring{\mathfrak{l}})$ *factors through*

$$\bar{z}^{-1/2}\Gamma(U, \Pi^*\check{S}) \subset \Gamma(U\backslash Z, \mathring{S}\otimes\mathring{\mathfrak{l}}). \qquad \blacksquare$$

Remark 4.25 (Polyhomogeneous expansion). Every $\psi \in \Gamma(U, \Pi^*\check{S})$ has a Taylor expansion

$$\psi \sim \sum_{k,\ell=0}^\infty \bar{z}^k z^\ell \check{\psi}_{k,\ell} \quad \text{with} \quad \check{\psi}_{k,\ell} \in \Pi^*\Gamma(Z, \check{S}\otimes_{\mathbf{C}} NZ^{k-\ell})$$

at Z. Here $\bar{z} \in \Gamma(U, \mathrm{Hom}_{\mathbf{C}}(\Pi^*NZ, \mathbf{C}))$ and $z \in \Gamma(U, \mathrm{Hom}_{\mathbf{C}}(\Pi^*NZ^{-1}, \mathbf{C}))$ denote the tautological sections. Therefore, by Corollary 4.24, every $\phi \in H_a^\infty\Gamma(X\backslash Z, S\otimes\mathfrak{l})$ has a polyhomogeneous expansion

$$\phi \sim \sum_{k,\ell=0}^\infty \bar{z}^{k-1/2} z^\ell \check{\phi}_{k,\ell} \quad \text{with} \quad \check{\phi}_{k,\ell} \in \Pi^*\Gamma(Z, \check{S}\otimes_{\mathbf{C}} NZ^{k-\ell})$$

at Z with $\bar{z}^{k-1/2} \coloneqq \bar{z}^k\bar{z}^{-1/2}$. Moreover, a moment's thought shows that if $D\phi = 0$, then the leading order term is of the form $\bar{z}^{k-1/2}\check{\phi}_{k,0}$ for some $k \in \mathbf{N}_0$. ♣

Corollary 4.26. *For every $k \in \mathbf{N}$ with $k \geqslant n/2$*

$$H_a^k\Gamma(X\backslash Z, S \otimes \mathfrak{l}) \subset r^{-1/2}L^\infty\Gamma(X\backslash Z, S \otimes \mathfrak{l}). \qquad \blacksquare$$

The above observation leads to the following "poor man's Weyl law".

Proposition 4.27 (Growth of eigenvalues). *Let $k \in \mathbf{N}_0$ with $k > n/2$. If $B \subset \check{H}\Gamma(X\backslash Z, S \otimes \mathfrak{l})$ is a $(k + \frac{1}{2})$–regular Lagrangian residue condition, then the* ***counting function*** *$N\colon [0, \infty) \to \mathbf{N}_0$ defined by*

$$N(\Lambda) \coloneqq \dim E_{\leqslant\Lambda} \quad \textit{with} \quad E_{\leqslant\Lambda} \coloneqq \bigoplus_{\lambda \in [-\Lambda,\Lambda]} \ker(D_B - \lambda \cdot \mathbf{1})$$

satisfies

$$N(\Lambda) \lesssim_{B,k} \langle\Lambda\rangle^{2k}.$$

Proof. The following argument is due to Li [Li80, Lemma 11]. Choose an L^2 orthonormal basis $(\phi_1, \ldots, \phi_{N(\Lambda)})$ of $E_{\leqslant\Lambda}$. The density $d \in C^\infty(X\backslash Z, [0, \infty)$ defined by

$$d \coloneqq \sum_{i=1}^{N(\Lambda)} |\phi_i|^2$$

does not depend on the choice of L^2 orthonormal basis. By construction

$$N(\Lambda) = \frac{1}{\operatorname{vol}(X)} \int_X d \lesssim \|rd\|_{L^\infty}.$$

Choose an $x \in X$ with $\|rd\|_{L^\infty} \leqslant 2|rd|(x)$. Since $\mathrm{ev}_x\colon E_{\leqslant\Lambda} \to (S \otimes \mathfrak{l})_x$ has rank at most $\operatorname{rk} S$, without loss of generality,

$$|rd|(x) = \sum_{i=1}^{\operatorname{rk} S} r|\phi_i|^2(x).$$

By Theorem 4.16 and Corollary 4.26

$$\|r^{1/2}\phi_i\|_{L^\infty} \lesssim \|\phi_i\|_{H_a^k} \lesssim_{B,k} \|D^k\phi_i\|_{L^2} + \|\phi_i\|_{L^2} \lesssim \langle\Lambda\rangle^k.$$

This implies the assertion. ■

Corollary 4.28. *Let $k \in \mathbf{N}_0$ with $k > n/2$. If $B \subset \check{H}\Gamma(X\backslash Z, S \otimes \mathfrak{l})$ is a $(k + \frac{1}{2})$–regular Lagrangian residue condition, then for every $t \in (0, \infty)$ the heat operator $h_t \coloneqq \exp(-tD_B^2)$ is trace class.* ■

Remark 4.29. Assume the situation of Corollary 4.28. If ε is a chirality operator, then for every $t > 0$

$$\operatorname{index} D_B^+ = \dim\ker D_B^+ - \dim\ker D_B^- = \operatorname{str}_\varepsilon h_t.$$

Here $\operatorname{str}_\varepsilon$ denotes the super trace with respect to ε of the heat operator h_t; cf. [BGV92, §1.3]. For suitable choices of B an analysis of the asymptotic behaviour of the kernel attached to h_t as $t \downarrow 0$ should result in index formulae analogous to the one established by Atiyah, Patodi, and Singer [APS75, Theorems 3.10 and 4.2]. It would be interesting to work this out in detail. Also, it should be mentioned that part of the unpublished PhD thesis [Yan07, Theorems 1.0.3 and 2.3.4] discusses such index formulae. ♣

4.5 Adapted Sobolev spaces, III: spectral description and tameness

Proposition 4.30. *The graded Fréchet space* $(H_a^\infty\Gamma(X\backslash Z, S \otimes \mathfrak{l}), (\|-\|_{H_a^k})_{k\in\mathbf{N}_0})$ *is tame.*

Proof. Consider the ∞–regular Lagrangian residue condition $B \coloneqq B_{\mathrm{APS}} \oplus L$ with $L \subset \ker A$ a Lagrangian subspace as discussed in Example 3.43. The operator D_B is self-adjoint and Fredholm, and Proposition 2.22 applies. The graded Fréchet space $(H_a^\infty\Gamma(X\backslash Z, S \otimes \mathfrak{l}; B), (\|-\|_{H_a^k})_{k\in\mathbf{N}_0})$ is tame in the sense of [Ham82, Part II Definition 1.3.2]. This can be seen as follows. Consider the graded Fréchet space $(\Sigma(L^2\Gamma(X\backslash Z, S \otimes \mathfrak{l})), (\|-\|_k)_{k\in\mathbf{N}_0})$ of exponentially decreasing sequences defined by

$$\Sigma(L^2\Gamma(X\backslash Z, S \otimes \mathfrak{l})) \coloneqq \left\{(\psi_\beta) \in L^2\Gamma(X\backslash Z, S \otimes \mathfrak{l})^{\mathbf{N}_0} : \|\psi_\beta\|_k < \infty \text{ for every } k \in \mathbf{N}_0\right\}$$

with

$$\|(\psi_\beta)\|_k^2 \coloneqq \sum_{\beta=0}^{\infty} e^{2k\beta}\|\psi_\beta\|_{L^2}^2;$$

cf. Hamilton [Ham82, Part II Example 1.1.4(b) with $q = 2$].

Let $(\phi_\alpha)_{\alpha\in\mathbf{N}_0}$ be an L^2 orthonormal basis of $L^2\Gamma(X\backslash Z, S\otimes\mathfrak{l})$ consisting of eigenspinors for D_B and denote by $(\lambda_\alpha)_{\alpha\in\mathbf{N}_0}$ the corresponding sequence of eigenvalues. Define $i\colon H_a^\infty\Gamma(X\backslash Z, S \otimes \mathfrak{l}; B) \to \Sigma(L^2\Gamma(X\backslash Z, S \otimes \mathfrak{l}))$ and $p\colon \Sigma(L^2\Gamma(X\backslash Z, S \otimes \mathfrak{l})) \to H_a^\infty\Gamma(X\backslash Z, S \otimes \mathfrak{l}; B)$ by

$$(i\phi)_\beta \coloneqq \sum_{\alpha=0}^{\infty} \mathbf{1}_{[e^\beta, e^{\beta+1})}(\langle\lambda_\alpha\rangle)\langle\phi, \phi_\alpha\rangle_{L^2}\phi_\alpha \quad \text{and} \quad p(\psi_\beta) \coloneqq \sum_{\alpha,\beta=0}^{\infty} \mathbf{1}_{[e^\beta, e^{\beta+1})}(\langle\lambda_\alpha\rangle)\langle\psi_\beta, \phi_\alpha\rangle_{L^2}\phi_\alpha.$$

A moment's thought shows that $p \circ i = \mathrm{id}$; moreover, by Theorem 4.16:

$$\|i(\phi)\|_k^2 = \sum_{\beta=0}^{\infty} e^{2k\beta}\|i(\phi)_\beta\|_{L^2}^2 \leqslant \sum_{\alpha=0}^{\infty}\langle\lambda_\alpha\rangle^{2k}\langle\phi, \phi_\alpha\rangle_{L^2}^2 \lesssim_k \|\phi\|_{H_a^k}$$

and

$$\begin{aligned}\|p(\psi_\beta)\|_{H_a^k}^2 &\lesssim_k \sum_{\alpha=0}^{\infty}\langle\lambda_\alpha\rangle^{2k}\langle p(\psi_\beta), \phi_\alpha\rangle^2 \\ &= \sum_{\alpha,\beta=0}^{\infty}\langle\lambda_\alpha\rangle^{2k}\mathbf{1}_{[e^\beta, e^{\beta+1})}(\langle\lambda_\alpha\rangle)\langle\psi_\beta, \phi_\alpha\rangle^2 \leqslant e^{2k}\sum_{\beta=0}^{\infty} e^{2k\beta}\|\psi_\beta\|_{L^2}^2.\end{aligned}$$

Therefore, $(H_a^\infty\Gamma(X\backslash Z, S\otimes\mathfrak{l}; B), (\|-\|_{H_a^k})_{k\in\mathbf{N}_0})$ is a tame direct summand in the sense of [Ham82, Part II Definition 1.3.1]) of $(\Sigma(L^2\Gamma(X\backslash Z, S \otimes \mathfrak{l})), (\|-\|_k)_{k\in\mathbf{N}_0})$.

A similar argument, using an L^2 orthonormal basis of JB consisting of eigenspinors for A in the definition of i and p, proves that the graded Fréchet space $(JB \cap \Gamma(Z, \check{S}), (\|-\|_{H^k})_{k\in\mathbf{N}_0})$ is a tame direct summand of $(\Sigma(L^2\Gamma(Z, \check{S})), (\|-\|_k)_{k\in\mathbf{N}_0})$.

Finally, by Lemma 4.11 and Theorem 4.10, $(H_a^\infty\Gamma(X\backslash Z, S \otimes \mathfrak{l}), (\|-\|_{H_a^k})_{k\in\mathbf{N}_0})$ is tamely isomorphic to

$$H_a^\infty\Gamma(X\backslash Z, S \otimes \mathfrak{l}) \cong H_a^\infty\Gamma(X\backslash Z, S \otimes \mathfrak{l}; B) \oplus (JB \cap \Gamma(Z, \check{S}))$$

and, therefore, by the above it is tame. ∎

4.6 Symbolic criterion for ∞–regularity

Proposition 4.31 (symbolic criterion for ∞–regularity). *Let $V \subset \check{S}$ be a subbundle. If*

$$\gamma(\xi)V \subset V^{\check{\Omega}} = JV^{\perp}$$

for every $\xi \in T^*Z\backslash\{0\}$, then B_V is ∞–regular.

Proof. The proof relies on the following observation which is already implicit in [FS98, Lemma 2]. Denote by $\mathrm{pr}_V : \check{S} \to \check{S}$ the orthogonal projection onto V. The operator $\mathrm{pr}_V A \mathrm{pr}_V$ is a 0–th order differential operator, because for every $f \in C^\infty(Z)$

$$[\mathrm{pr}_V A \mathrm{pr}_V, f] = -\mathrm{pr}_V J\gamma(\mathrm{d}f)\mathrm{pr}_V = 0.$$

Let $k \in \mathbf{N}_0$. Since $\mathrm{pr}_V A \mathrm{pr}_V$ is a 0–th order differential operator, $\mathrm{pr}_V A^{2k+1}\mathrm{pr}_V$ is a $2k$–th order differential operator. Therefore, for every $\phi \in \Gamma(Z, V)$ and $\varepsilon > 0$

$$\begin{aligned}\langle A^{2k+1}\phi, \phi\rangle_{L^2} = \langle \mathrm{pr}_V A^{2k+1}\mathrm{pr}_V \phi, \phi\rangle_{L^2} &\lesssim_{B_V,k} \|\phi\|^2_{H^k} \\ &\lesssim \|\phi\|_{H^{k-1/2}}\|\phi\|_{H^{k+1/2}} \lesssim \varepsilon^{-1}\|\phi\|^2_{H^{k-1/2}} + \varepsilon\|\phi\|^2_{H^{k+1/2}};\end{aligned}$$

moreover, by direct inspection,

$$\langle A^{2k+1}\phi, \phi\rangle_{L^2} = -\||A|^{k+1/2}\mathbf{1}_{(-\infty,0)}(A)\phi\|^2_{L^2} + \||A|^{k+1/2}\mathbf{1}_{[0,\infty)}(A)\phi\|^2_{L^2}.$$

As a consequence, for every $\phi \in B_V$,

$$\|\phi\|^2_{H^{k+1/2}} \lesssim_{B,k} \|\mathbf{1}_{(-\infty,0)}(A)\phi\|^2_{H^{k+1/2}} + \|\phi\|^2_{H^{k-1/2}}.$$

By induction, B_V is $(k + 1/2)$–regular for every $k \in \mathbf{N}_0$; hence: ∞–regular. ∎

Corollary 4.32. *Let $V \subset \check{S}$ be a subbundle. If*

$$\gamma(\xi)V = JV^{\perp}$$

for every $\xi \in T^*Z\backslash\{0\}$, then B_V and B_V^G are ∞–regular and Fredholm. ∎

Example 4.33. The local residue conditions defined in Example 3.46 satisfy the criterion in Corollary 4.32. ♠

Example 4.34. If

$$\mathbf{L} \in \Gamma(Z, \mathrm{Hom}_{\mathbf{C}}(\overline{NZ}, \check{S}))$$

is nowhere-vanishing, then

$$V \coloneqq \mathrm{im}\,\mathbf{L} \subset \check{S}$$

is a rank one complex subbundle. Therefore and since $J\gamma(\xi)$ and $IJ\gamma(\xi)$ are skew-adjoint, $\gamma(\xi)V \subseteq JV^{\perp}$ for every $\xi \in T^*Z\backslash\{0\}$; that is: V satisfies the criterion in Proposition 4.31. Moreover: if $\mathrm{rk}_{\mathbf{C}}\,\check{S} = 2$, then $J\gamma(\xi)V = V^{\perp}$ and B_V is self-adjoint. ♠

Remark 4.35. Suppose that $(Z, \mathfrak{l}; \Phi)$ is a $\mathbf{Z}/2\mathbf{Z}$ harmonic spinor whose branching locus Z satisfies Hypothesis 3.1. Since $\Phi \in \ker D_{\min}$, by Remark 4.25, the polyhomogeneous expansion of Φ at Z is of the form

$$\Phi \sim \bar{z}^{1/2}\check{\Phi}_{1,0} + \sum_{\substack{k,\ell=0 \\ k+\ell\geqslant 2}}^{\infty} \bar{z}^{k-1/2} z^{\ell} \check{\Phi}_{k,\ell}.$$

The leading coefficient $\check{\Phi}_{1,0}$ determines an $\mathbf{L}_\Phi \in \Gamma(Z, \mathrm{Hom}_{\mathbf{C}}(\overline{NZ}, \check{S}))$. If $\operatorname{rk} S = 4$ and $\check{\Phi}_{1,0}$ is nowhere vanishing, then this produces an ∞–regular Lagrangian local residue condition. This is the residue condition behind the scenes in [Tak15; Par23]. ♣

Remark 4.36. In the presence of a chirality operator ε the above discussion refines as follows:

(1) Let $V^+ \subset \check{S}^+$ be subbundle. If

$$\gamma(\xi)V^+ = J(V^+)^\perp \cap \check{S}^+$$

for every $\xi \in T^*Z\backslash\{0\}$, then $V := V^+ \oplus V^-$ with $V^- := J(V^+)^\perp \cap \check{S}^-$ satisfies the condition in Corollary 4.32.

(2) If $\operatorname{rk} S^+ = 4$ and

$$\mathbf{L} \in \Gamma(Z, \mathrm{Hom}_{\mathbf{C}}(\overline{NZ}, \check{S}^+))$$

is nowhere vanishing, then $V^+ := \operatorname{im} \mathbf{L}$ satisfies the above condition.

(3) As in Remark 4.35, the leading coefficient of the polyhomogeneous expansion of a positive $\mathbf{Z}/2\mathbf{Z}$ harmonic spinor Φ determines a $\mathbf{L}_\Phi \in \Gamma(Z, \mathrm{Hom}_{\mathbf{C}}(\overline{NZ}, \check{S}^+))$. ♣

A Non-coorientable branching loci

The following discussion explains what changes need to be made in Section 3 and Section 4 if $Z \subset X$ is a closed submanifold of codimension two, but not cooriented or even coorientable.

The **coorientation bundle**

$$\mathfrak{o} := \Lambda^2 NZ \to Z$$

is a Euclidean line bundle and its unique orthogonal connection is flat. The Euclidean metric on NZ identifies $\mathfrak{o}$ with the bundle of skew-adjoint endomorphisms of NZ. The evaluation map defines a isometry

$$I\colon\ \mathfrak{o} \otimes NZ \to NZ.$$

A moment's thought shows that the diagram

$$\begin{tikzcd}
NZ \arrow[r, "\cong"] \arrow[rrr, bend right, "-\mathbf{1}"'] & \mathfrak{o}^{\otimes 2} \otimes NZ \arrow[r, "\mathbf{1}\otimes I"] & \mathfrak{o} \otimes NZ \arrow[r, "I"] & NZ
\end{tikzcd}$$

commutes. A trivialisation $\mathfrak{o} \cong \underline{\mathbf{R}}$ enhances I to an orthogonal almost complex structure on NZ; that is: a coorientation of Z enhances NZ to a Hermitian line bundle.

The canonical isomorphism $\mathfrak{o}^{\otimes 2} \cong \mathbf{R}$ produces the flat bundle

$$\mathbf{A} \coloneqq \mathbf{R} \oplus i \cdot \mathfrak{o} \to Z$$

of normed $\mathbf{R}$–algebras whose fibres are isomorphic to $\mathbf{C}$, canonically up to complex conjugation. The above discussion reveals NZ to be a bundle of Euclidean $\mathbf{A}$–modules of rank one. Systematically replacing $\mathbf{C}$ by $\mathbf{A}$ and tracking the use of $\mathfrak{o}$ in Section 3 and Section 4 removes the need for a coorientation of Z:

- The frame bundle $\pi\colon F \to Z$ defined in Definition 3.4 is not $\mathrm{U}(1)$–principal. Its vertical tangent bundle $\ker T\pi$ is canonically isomorphic to $i\pi^*\mathfrak{o}$. Therefore, the Levi-Civita connection defines $i\theta \in \Omega^1(F, i\pi^*\mathfrak{o})$; moreover, $\partial_\alpha \in \Gamma(F, \pi^*(\mathfrak{o} \otimes NZ))$.

- Definition 3.8 reveals $S|_Z$ to be an $\mathbf{A}$–module and defines $J \in \Gamma(F, \mathrm{End}(\underline{S}))$ and $I, K = IJ \in \Gamma(F, \pi^*\mathfrak{o} \otimes \mathrm{End}(\underline{S}))$. The sign ambiguities in the term $I\mathring{\nabla}_{\partial_\alpha}$ appearing in Remark 3.13, Proposition 3.22, and the proof of Lemma 4.23 cancel.

- Definition 3.19 constructs NZ^λ as an $\mathbf{A}$–module. Remark 3.20, Proposition 3.21, Proposition 3.22 hold with $\mathbf{C}$ replaced by $\mathbf{A}$. This can be seen, e.g., by passing to the double cover $\tilde{Z} \to Z$ defined by $\mathfrak{o}$.

- In the definition of the residue bundle $\check{S}$ and the branching locus operator A in Definition 3.36 the appearances of $\mathbf{C}$ need to be replaced by $\mathbf{A}$. $\check{S}$ inherits $J \in \Gamma(Z, \mathrm{End}(\check{S}))$ and $I, K = IJ \in \Gamma(Z, \mathfrak{o} \otimes \mathrm{End}(\check{S}))$.

- $\mathbf{C}$ needs to be replaced by $\mathbf{A}$ in Example 3.46, Definition 4.22, Remark 4.25.

References

[AG16] Pierre Albin and Jesse Gell-Redman. *The index of Dirac operators on incomplete edge spaces. SIGMA Symmetry Integrability Geom. Methods Appl.* 12 (2016), Paper No. 089, 45. DOI: 10.3842/SIGMA.2016.089. ☜ (cit. on p. 4)

[AG23] Pierre Albin and Jesse Gell-Redman. *The index formula for families of Dirac type operators on pseudomanifolds. J. Differential Geom.* 125.2 (2023), pp. 207–343. DOI: 10.4310/jdg/1696432923. ☜ (cit. on p. 4)

[APS75] Michael F. Atiyah, V. K. Patodi, and Isadore M. Singer. *Spectral asymmetry and Riemannian geometry. I. Mathematical Proceedings of the Cambridge Philosophical Society* 77 (1975), pp. 43–69. MR: 0397797. ☜ (cit. on pp. 3, 27, 38)

[BB12] Christian Bär and Werner Ballmann. *Boundary value problems for elliptic differential operators of first order. Surveys in differential geometry.* 17. International Press, 2012, pp. 1–78. DOI: 10.4310/SDG.2012.v17.n1.a1. arXiv: 1101.1196. MR: 3076058. Zbl: 1331.58022 (cit. on pp. 3, 28, 31)

[BGV92] Nicole Berline, Ezra Getzler, and Michèle Vergne. *Heat kernels and Dirac operators.* Grundlehren der Mathematischen Wissenschaften 298. Springer-Verlag, 1992. DOI: 10.1007/978-3-642-58088-8. MR: 1215720. Zbl: 0744.58001 (cit. on p. 38)

[Bis89] Jean-Michel Bismut. *A local index theorem for non-Kähler manifolds.* English. *Math. Ann.* 284.4 (1989), pp. 681–699. DOI: 10.1007/BF01443359. Zbl: 0666.58042 (cit. on pp. 2, 5)

[BF98] Bernhelm Booß-Bavnbek and Kenro Furutani. *The Maslov index: A functional analytical definition and the spectral flow formula.* English. *Tokyo J. Math.* 21.1 (1998), pp. 1–34. DOI: 10.3836/tjm/1270041982. Zbl: 0932.37063 (cit. on pp. 3, 4, 7, 8)

[BS88] Jochen Brüning and Robert Seeley. *An index theorem for first order regular singular operators. Amer. J. Math.* 110.4 (1988), pp. 659–714. DOI: 10.2307/2374646. MR: 955293 (cit. on p. 22)

[BS18] Theo Bühler and Dietmar A. Salamon. *Functional analysis.* Vol. 191. Graduate Studies in Mathematics. American Mathematical Society, 2018. DOI: 10.1090/gsm/191. MR: 3823238 (cit. on pp. 3, 4, 6, 9)

[DW24] Aleksander Doan and Thomas Walpuski. *Chambered invariants of real Cauchy–Riemann operators.* 2024. arXiv: 2410.21057 (cit. on pp. 6, 22, 35)

[Don17] Simon K. Donaldson. *Adiabatic limits of co-associative Kovalev–Lefschetz fibrations. Algebra, Geometry, and Physics in the 21st Century.* Springer, 2017, pp. 1–29. DOI: 10.1007/978-3-319-59939-7_1. arXiv: 1603.08391. Zbl: 1409.53044 (cit. on pp. 2, 3)

[Don21] Simon K. Donaldson. *Deformations of multivalued harmonic functions. The Quarterly Journal of Mathematics* 72.1-2 (2021), pp. 199–235. DOI: 10.1093/qmath/haab018 (cit. on p. 3)

[FS98] Simone Farinelli and Günter Schwarz. *On the spectrum of the Dirac operator under boundary conditions. J. Geom. Phys.* 28.1-2 (1998), pp. 67–84. DOI: 10.1016/S0393-0440(98)00013-8. (cit. on p. 40)

[Ham82] Richard S. Hamilton. *The inverse function theorem of Nash and Moser. Bull. Amer. Math. Soc. (N.S.)* 7.1 (1982), pp. 65–222. DOI: 10.1090/S0273-0979-1982-15004-2. (cit. on pp. 4, 39)

[HMT23] Andriy Haydys, Rafe Mazzeo, and Ryosuke Takahashi. *An index theorem for* $\mathbb{Z}/2$*–harmonic spinors branching along a graph.* 2023. arXiv: 2310.15295 (cit. on pp. 4, 6)

[He22] Siqi He. *The branched deformations of the special Lagrangian submanifolds.* 2022. DOI: 10.48550/ARXIV.2202.12282. arXiv: 2202.12282 (cit. on p. 3)

[Joh75] Kenneth A. Johnson. *The M.I.T. Bag Model. Acta Phys. Polon. B* 6 (1975), p. 865 (cit. on p. 28)

[LM89] H. Blaine Lawson Jr. and Marie-Louise Michelsohn. *Spin geometry.* Princeton Mathematical Series 38. Princeton University Press, 1989. MR: 1031992. Zbl: 0688.57001 (cit. on p. 2)

[Li80] Peter Li. *On the Sobolev constant and the* p*–spectrum of a compact Riemannian manifold. Annales Scientifiques de l'École Normale Supérieure* 13 (1980), pp. 451–468. Zbl: 0466.53023 (cit. on p. 38)

[Maz91] Rafe Mazzeo. *Elliptic theory of differential edge operators. I. Communications in Partial Differential Equations* 16.10 (1991), pp. 1615–1664. DOI: 10.1080/03605309108820815. Zbl: 0745.58045 (cit. on p. 4)

[MV14] Rafe Mazzeo and Boris Vertman. *Elliptic theory of differential edge operators, II: Boundary value problems. Indiana Univ. Math. J.* 63.6 (2014), pp. 1911–1955. DOI: 10.1512/iumj.2014.63.5435. 🔓 (cit. on p. 4)

[MS98] Dusa McDuff and Dietmar A. Salamon. *Introduction to symplectic topology.* 2nd edition. Oxford Mathematical Monographs. Oxford University Press, 1998. MR: 1698616. Zbl: 1066.53137 (cit. on pp. 3, 4)

[Mel93] Richard B. Melrose. *The Atiyah–Patodi–Singer index theorem.* Vol. 4. Research Notes in Mathematics. A. K. Peters, 1993. DOI: 10.1016/0377-0257(93)80040-i. MR: 1348401. Zbl: 0796.58050 (cit. on p. 31)

[Par23] Gregory J. Parker. *Deformations of* $\mathbf{Z}_2$*–Harmonic Spinors on* 3*–Manifolds.* 2023. arXiv: 2301.06245 (cit. on pp. 3, 4, 41)

[PSS18a] Fabian Portmann, Jérémy Sok, and Jan Philip Solovej. *Analysis of zero modes for Dirac operators with magnetic links. J. Funct. Anal.* 275.3 (2018), pp. 604–659. DOI: 10.1016/j.jfa.2017.12.006. MR: 3806820 (cit. on p. 4)

[PSS18b] Fabian Portmann, Jérémy Sok, and Jan Philip Solovej. *Self-adjointness and spectral properties of Dirac operators with magnetic links. J. Math. Pures Appl. (9)* 119 (2018), pp. 114–157. DOI: 10.1016/j.matpur.2017.10.010. MR: 3862145 (cit. on p. 4)

[PSS20] Fabian Portmann, Jérémy Sok, and Jan Philip Solovej. *Spectral flow for Dirac operators with magnetic links. J. Geom. Anal.* 30.1 (2020), pp. 1100–1167. DOI: 10.1007/s12220-018-00128-5. 🔓 (cit. on p. 4)

[RS75] Michael Reed and Barry Simon. *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness.* Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1975. MR: 493420 (cit. on p. 9)

[SW08] Dietmar A. Salamon and Katrin Wehrheim. *Instanton Floer homology with Lagrangian boundary conditions. Geometry and Topology* 12.2 (2008), pp. 747–918. DOI: 10.2140/gt.2008.12.747. MR: 2403800 (cit. on pp. 3, 4, 8, 10)

[Tak15] Ryosuke Takahashi. *The moduli space of* S^1*–type zero loci for* $\mathbf{Z}/2$*–harmonic spinors in dimension* 3. 2015. arXiv: 1503.00767 (cit. on pp. 3, 4, 13, 41)

[Tak17] Ryosuke Takahashi. *Index theorem for the zero loci of* $\mathbf{Z}/2$*–harmonic spinors.* 2017. arXiv: 1705.01954 (cit. on p. 3)

[Tau13a] Clifford H. Taubes. $\mathbf{PSL}(2;\mathbf{C})$ *connections on 3–manifolds with* L^2 *bounds on curvature. Cambridge Journal of Mathematics* 1.2 (2013), pp. 239–397. DOI: 10.4310/CJM.2013.v1.n2.a2. arXiv: 1205.0514. MR: 3272050. Zbl: 1296.53051 (cit. on p. 2)

[Tau13b] Clifford H. Taubes. *Compactness theorems for* $\mathrm{SL}(2;\mathbf{C})$ *generalizations of the* 4*–dimensional anti-self dual equations.* 2013. arXiv: 1307.6447 (cit. on p. 2)

[Tau16] Clifford H. Taubes. *On the behavior of sequences of solutions to* $\mathrm{U}(1)$ *Seiberg–Witten systems in dimension* 4. 2016. arXiv: 1610.07163 (cit. on p. 2)

[Tau17] Clifford H. Taubes. *The behavior of sequences of solutions to the Vafa–Witten equations.* 2017. arXiv: 1702.04610 (cit. on p. 2)

[Tau22] Clifford H. Taubes. *Sequences of Nahm pole solutions to the* $\mathrm{SU}(2)$ *Kapustin–Witten equations.* 2022 (cit. on p. 2)

[vNeu30] John von Neumann. *Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren. Math. Ann.* 102.1 (1930), pp. 49–131. DOI: 10.1007/BF01782338. MR: 1512569 (cit. on p. 9)

[Yan07] Fangyun Yang. *Dirac operators and monopoles with singularities.* PhD thesis. Massachusetts Institute of Technology, 2007. MR: 2717453. ☞ (cit. on p. 38)