Arithmetic conditions for the existence of G_2–instantons over twisted connected sums

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Abstract

Extending earlier work in [Wal16] this article introduces an arithmetic condition which guarantees the existence of G_2–instantons over twisted connected sums. By brute-force search a significant number of solutions of this condition can be found. This yields many new examples of G_2–instantons and, in particular, the first examples of irreducible, unobstructed G_2–instantons on $\mathbf{P}\mathrm{U}(r)$–bundles for $r \neq 2$.

1 Introduction

The first few examples examples of irreducible unobstructed G_2–instantons on SO(3)–bundles where constructed in [Wal13]. These examples are defined over G_2–manifolds constructed by Joyce [Joy96a; Joy96b] by resolving flat G_2–orbifolds. By far the most fruitful method for constructing G_2–manifolds to date is the twisted connected sum construction [Kov03; KL11; CHNP13; CHNP15]. While there is a gluing theorem to produce G_2–instantons over twisted connected sums [SW15], so far there are only two examples of G_2–instantons constructed using this theorem in the literature [Wal16; MNS17]. This article slightly extends the work in [Wal16] and shows that the ideas developed there can, in fact, be used to produce a rather large number of G_2–instantons.

After reviewing (a special case of) the twisted connected sum construction in Section 2, an arithmetic condition for the existence of G_2–instantons is given and proved in Section 3. Solutions to this arithmetic condition can be found by a simple brute-force search algorithm outlined in Section 4. A concrete implementation in Sage/Python of this algorithm (with a quite restricted search scope) finds 299 solutions of the aforementioned arithmetic condition. This yields, in particular, the first examples of irreducible, unobstructed G_2–instantons on $\mathbf{P}\mathrm{U}(r)$–bundles with $r \neq 2$. Further statistics regarding these can be found in Section 5.

2 Twisted connected sums from Fano 3–folds

2.1 The twisted connected sum construction

Definition 2.1 (Corti, Haskins, Nordström, and Pacini [CHNP13, Definition 5.1]). A **building block** is a smooth projective 3–fold Z together with a projective morphism $\pi\colon Z \to \mathbf{P}^1$ such that the following hold:

1. The anticanonical class $-K_Z \in H^2(Z)$ is primitive.
2. $\Sigma := \pi^*(\infty)$ is a smooth $K3$ surface and $\Sigma \sim -K_Z$.

Definition 2.2. A **framing** of a building block (Z, Σ) consists of a hyperkähler structure $\boldsymbol{\omega} = (\omega_I, \omega_J, \omega_K)$ on Σ such that $\omega_J + i\omega_K$ is of type $(2,0)$ as well as a Kähler class on Z whose restriction to Σ is $[\omega_I]$.

Given a framed building block $(Z, \Sigma, \boldsymbol{\omega})$, using the work of Haskins, Hein, and Nordström [HHN15], we can make $V := Z\backslash\Sigma$ into an asymptotically cylindrical (ACyl) Calabi–Yau 3–fold with asymptotic cross-section $S^1 \times \Sigma$; hence, $Y := S^1 \times V$ is an ACyl G_2–manifold with asymptotic cross-section $T^2 \times \Sigma$.

Definition 2.3. A **matching** of pair of framed building blocks $(Z_\pm, \pi_\pm, \boldsymbol{\omega}_\pm)$ is a **hyperkähler rotation** $\mathfrak{r}\colon \Sigma_+ \to \Sigma_-$, i.e., a diffeomorphism such that

$$\mathfrak{r}^*\omega_{I,-} = \omega_{J,+}, \quad \mathfrak{r}^*\omega_{J,-} = \omega_{I,+} \quad \text{and} \quad \mathfrak{r}^*\omega_{K,-} = -\omega_{K,+}.$$

Given a matched pair of framed building blocks $(Z_\pm, \pi_\pm, \boldsymbol{\omega}_\pm; \mathfrak{r})$, the **twisted connected sum** construction produces a simply-connected compact 7–manifold Y together with a family of torsion-free G_2–structures $\{\phi_T : T \gg 1\}$ by gluing truncations of $Y_\pm$ along their boundaries via interchanging the circle factors and $\mathfrak{r}$. Denote by $\Upsilon\colon H^{\mathrm{ev}}(Z_+) \times_{H^{\mathrm{ev}}(\Sigma_+)} H^{\mathrm{ev}}(Z_-) \to H^{\mathrm{ev}}(Y)$ the splicing map defined in [CHNP15, Definition 4.15].

2.2 Building blocks from Fano 3–folds

Proposition 2.4 ([Kov03, Proposition 6.42; CHNP15, Proposition 3.17]). *Let W be a Fano 3–fold and let $|\Sigma_0, \Sigma_\infty| \in |-K_W|$ be anti-canonical pencil such that Σ_∞ is a smooth K3 surface and the base locus C is a smooth curve. Set*

$$Z := \mathrm{Bl}_C W$$

and denote by $\pi\colon Z \to \mathbf{P}^1$ the map induced by the pencil $|\Sigma_0, \Sigma_\infty|$. In this situation the following hold:

1. *$\pi\colon Z \to \mathbf{P}^1$ is a building block and*
2. *the inclusion $\Sigma_\infty \subset W$ induces an isomorphism $\pi^*(\infty) \cong \Sigma_\infty$.*

Definition 2.5. A building block arising from Proposition 2.4 is said to be of **Fano type**.

Remark 2.6. Let W be a Fano 3–fold whose anticanonical bundle $-K_W$ is very ample. By Bertini's theorem, the adjunction formula, and the Lefschetz hyperplane theorem a general anti-canonical divisor is a smooth K3 surface Σ_∞. A further application of Bertini's theorem shows that having chosen Σ_∞ one can always find Σ_0 such that the base locus of $|\Sigma_0, \Sigma_\infty|$ is smooth. Moreover, for a general Σ_0 the morphism π has only finitely many singular fibres and the singular fibres have only ordinary double point singularities; see [Voi07, Corollary 2.10].

2.3 Matching Fano 3–folds

Definition 2.7. The **Picard lattice** of a smooth complex 3–fold W is $\mathrm{Pic}(W)$ equipped with the quadratic form

$$x \otimes y \mapsto x \cdot y \cdot (-K_W).$$

Definition 2.8. Let N be a lattice. An N**–marking** of a Fano 3–fold W is an isometry $h\colon N \longrightarrow \mathrm{Pic}(W)$. A pair of N–marked Fano 3–folds (W_1, h_1) and (W_2, h_2) are **deformation equivalent** if there is a proper holomorphic submersion $\pi\colon X \longrightarrow \Delta$ from a complex manifold to the unit disc in $\mathbf{C}$ such that W_1 and W_2 both occur as fibers of π and parallel transport induces the isometry $h_1^{-1}h_2$. An N**–marked deformation type** of Fano 3–folds is a maximal set $\mathscr{W}$ of N–marked Fano 3–folds such that every pair of elements of $\mathscr{W}$ are deformation equivalent.

Definition 2.9. Let $N_\pm$ and N_0 be non-degenerate lattices and let $i_\pm\colon N_0 \hookrightarrow N_\pm$ be embeddings. An **orthogonal pushout** of i_+ and i_- is a lattice N together with embeddings $j_\pm\colon N_\pm \hookrightarrow N$ such that the diagram

$$\begin{tikzcd}[column sep=small] & N_+ \arrow[dr, "j_+"] & \\ N_0 \arrow[ur, "i_+"] \arrow[dr, "i_-"'] & & N \\ & N_- \arrow[ur, "j_-"'] & \end{tikzcd}$$

is commutative,

$$j_+(N_+) \cap j_-(N_-) = i_\pm j_\pm(N_0), \quad N = j_+(N_+) + j_-(N_-), \quad \text{and} \quad j_\pm(N_\pm)^\perp \subset j_\mp(N_\mp).$$

Situation 2.10. *Let $r_\pm, r_0 \in \mathbf{N}$ with $r_+ + r_- - r_0 \leqslant 11$. Let $N_\pm$ be lattices of signature $(1, r_\pm - 1)$, let N_0 be a lattice of signature $(0, r_0)$, let $i_\pm\colon N_0 \hookrightarrow N_\pm$ be primitive embeddings, and let (N, j_+, j_-) be an orthogonal pushout of i_+ and i_-. Let $\mathrm{Amp}_\pm \subset N_\pm \otimes_{\mathbf{Z}} \mathbf{R}$ be open cones. Let $\mathscr{W}_\pm$ be $N_\pm$–marked deformation types of Fano 3–folds such that for every $(W_\pm, h_\pm) \in \mathscr{W}$:*

1. *$-K_{W_\pm}$ is very ample,*
2. *$h_\pm^{-1}(\mathrm{Amp}_\pm)$ is contained in the ample cone of $W_\pm$, and*
3. *$\mathrm{Amp}_\pm \cap (i_\pm(N_0) \otimes_{\mathbf{Z}} \mathbf{R})^\perp \neq \varnothing$.*

Proposition 2.11. *Assume Situation 2.10. For every $H_\pm \in \mathrm{Amp}_\pm \cap (i_\pm(N_0) \otimes_{\mathbf{Z}} \mathbf{R})^\perp$ and $\varepsilon > 0$, there exist $(W_\pm, h_\pm) \in \mathscr{W}_\pm$, smooth K3 surfaces $\Sigma_\pm \in |-K_{W_\pm}|$, hyperkähler structures $\boldsymbol{\omega}^\pm = (\omega_I^\pm, \omega_J^\pm, \omega_K^\pm)$ on $\Sigma_\pm$, and a hyperkähler rotation $\mathfrak{r}\colon (\Sigma_+, \boldsymbol{\omega}_+) \to (\Sigma_-, \boldsymbol{\omega}_-)$ such that:*

1. *the restriction maps $\mathrm{res}_\pm\colon \mathrm{Pic}(W_\pm) \to \mathrm{Pic}(\Sigma_\pm)$ are isomorphisms,*

2. *the diagram*

$$\begin{tikzcd} & \mathrm{Pic}(W_+) \arrow[r, "\mathrm{res}_+"] & \mathrm{Pic}(\Sigma_+) \arrow[d, "\mathfrak{r}_*"] \\ N_0 \arrow[ur, "h_+"] \arrow[dr, "h_+"'] & & \\ & \mathrm{Pic}(W_-) \arrow[r, "\mathrm{res}_-"'] & \mathrm{Pic}(\Sigma_-) \end{tikzcd}$$

   is commutative, and

3. *the distance between $\mathbf{R}h_\pm(H_\pm)$ and $\mathbf{R}[\omega_I^\pm]$ in $\mathbf{P}H^2(\Sigma_\pm, \mathbf{R})$ is at most ε.*

Proof. Therefore, [CHNP15, Proposition 6.18] applies. By [Moĭ67, Theorem 7.5] for very general $\Sigma_\pm \in -K_{W_\pm}$ the conclusion (1) holds. By [Nik79, Theorem 1.12.4 and Corollary 1.12.3], N embeds primitively into the K3 lattice. Therefore, the framing and hyperkähler rotation can thus be obtained from [CHNP15, Proposition 6.18] and (2) holds by construction. The observation that in this construction one may assume (3) is due to [MNS17, Proposition 2.6]. □

Choosing further anticanonical divisors as in Proposition 2.4 one obtains a matched pair of framed building blocks $(Z_\pm, \pi_\pm, \boldsymbol{\omega}_\pm; \mathfrak{r})$ which can be feed into the twisted connected sum construction.

3 An existence theorem for G_2–instantons

Theorem 3.1. *Assume Situation 2.10. Let*

$$H_\pm \in \mathrm{Amp}_\pm \cap i_\pm(N_0)^\perp \quad \text{and} \quad v = (r, B, s) \in \{2, 3, \ldots\} \times N_0 \times \mathbf{Z}$$

such that the following hold:

1. $B^2 = 2(rs - 1)$,

2. *the r and the divisibility of B are coprime, r and s are coprime; and*

3. *for every non-zero $x \in N_\pm$ perpendicular to $H_\pm$*

$$x^2 < -\frac{1}{2}r^2(r^2 - 1).$$

In this situation there is a matched pair of framed building blocks $(Z_\pm, \pi_\pm, \boldsymbol{\omega}_\pm; \mathfrak{r})$ obtained from Proposition 2.11 and for $T \gg 1$ the corresponding twisted connected sum carries an irreducible and unobstructed G_2–instanton on a non-trivial $\mathbf{PU}(r)$–bundle.

3.1 A gluing theorem for G_2–instantons over twisted connected sums

Theorem 3.2 (Sá Earp and Walpuski [SW15, Theorem 1.3 and Remark 1.7]). *Let $(Z_\pm, \pi_\pm, \boldsymbol{\omega}_\pm; \mathfrak{r})$ be a matched pair of framed building blocks. Set $\Sigma_\pm := \pi_\pm^*(\infty)$. Denote by Y the compact 7–manifold and by $\{\phi_T : T \gg 1\}$ the family of torsion-free G_2–structures obtained from the twisted connected sum construction. Let $\mathscr{E}_\pm \to Z_\pm$ be a pair of rank r holomorphic vector bundles such that the following hold:*

1. $c_1(\mathscr{E}_+|_{\Sigma_+}) = \mathfrak{r}^* c_1(\mathscr{E}_-|_{\Sigma_-})$ *and* $c_2(\mathscr{E}_+|_{\Sigma_+}) = \mathfrak{r}^* c_2(\mathscr{E}_-|_{\Sigma_-})$.
2. $\mathscr{E}_\pm|_{\Sigma_\pm}$ *is μ–stable with respect to $\omega_{I,\pm}$ and rigid, i.e.,*
$$H^1(\Sigma_\pm, \mathscr{E}\mathrm{nd}_0(\mathscr{E}_\pm|_{\Sigma_\pm})) = 0.$$
3. $\mathscr{E}_\pm$ *is infinitesimally rigid:*
$$H^1(Z_\pm, \mathscr{E}\mathrm{nd}_0(\mathscr{E}_\pm)) = 0.$$

In this situation exists a $\mathrm{U}(r)$–bundle E over Y with

$$c_1(E) = \Upsilon(c_1(\mathscr{E}_+), c_1(\mathscr{E}_-)) \quad \textit{and} \quad c_2(E) = \Upsilon(c_2(\mathscr{E}_+), c_2(\mathscr{E}_-)).$$

and a family of connections $\{A_T : T \gg 1\}$ on the associated $\mathbf{PU}(r)$–bundle with A_T being an irreducible unobstructed G_2–instanton over (Y, ϕ_T).

3.2 Relative moduli spaces of sheaves

Definition 3.3. Let $\pi\colon Z \to \mathbf{P}^1$ be a building block. Define $\mathbf{M}\colon \mathbf{Sch}_{\mathbf{P}^1}^{\mathrm{op}} \to \mathbf{Set}$, the **relative moduli functor** of coherent sheaves on Z, as follows:

- Let S be $\mathbf{P}^1$–scheme. Two such sheaves $\mathscr{E}$ and $\mathscr{F}$ over $Z \times_{\mathbf{P}^1} S$ are consider to be equivalent if there exists a line bundle over S such that $\mathscr{E}$ and $\mathscr{F} \otimes \mathscr{L}$ are isomorphic. We define $\mathbf{M}(S)$ the be the set of equivalence classes of S–flat coherent sheaves on $Z \times_{\mathbf{P}^1} S$.
- Given a morphism $f\colon S \to T$ of $\mathbf{P}^1$–schemes, set
$$\mathbf{M}(f)[\mathscr{E}] := [f^*\mathscr{E}].$$

Definition 3.4. Let $\pi\colon Z \to \mathbf{P}^1$ be a building block. Let $\clubsuit$ denote an open condition on coherent sheaves on the fibers of π. Denote by $\mathbf{M}_\clubsuit$ the open subfunctor of $\mathbf{M}$ defined by

$$\mathbf{M}_\clubsuit(S) := \{[\mathscr{E}] \in \mathbf{M}(S) : \text{for every } b \in S \text{ the fiber } \mathscr{E} \otimes_{\mathscr{O}_S} \mathbf{C}(b) \text{ satisfies } \clubsuit\}.$$

We say that $\mathbf{M}_\clubsuit$ is **representable** if there exists a $\mathbf{P}^1$–scheme $M_\clubsuit$ together with a natural isomorphism

$$\phi\colon \mathrm{Hom}_{\mathbf{P}^1}(\cdot, M_\clubsuit) \cong \mathbf{M}_\clubsuit(\cdot).$$

In this case we call $M_\clubsuit$ the **relative moduli space** of coherent sheaves on Z satisfying $\clubsuit$. A **universal sheaf** over $Z \times_{P^1} M_\clubsuit$ is a sheaf in the equivalence class $\phi(\mathrm{id}_{M_\clubsuit}) \in \mathbf{M}_\clubsuit(M_\clubsuit)$.

Remark 3.5. Typical examples of representable subfunctors of $\mathbf{M}$ arise by imposing stability conditions [Mar78; Mar77; Sim94]. However, these are not the only examples; see, e.g., Drézet [Dré08].

Suppose that $M = M_\clubsuit$ is a relative moduli space of coherent sheaves on Z with universal sheaf $\mathscr{U}$. Being a $\mathbf{P}^1$–scheme, M comes with a morphism $\varpi\colon M \to \mathbf{P}^1$. A morphism $\mathbf{P}^1 \to M$ of $\mathbf{P}^1$–schemes is simply a morphism $s\colon \mathbf{P}^1 \to M$ with $\varpi \circ s = \mathrm{id}_{\mathbf{P}^1}$, that is, a section of ϖ. By construction any such section yields a coherent sheaf $(\mathrm{id}_Z \times_{\mathbf{P}^1} s)^*\mathscr{U}$ on Z.

Proposition 3.6. *Let $\pi\colon Z \to \mathbf{P}^1$ be a building block and let A be a π–ample line bundle. Let $v = (r, B, s) \in \mathbf{N}_0 \times H^2(Z) \times \mathbf{Z}$. Suppose that r and s are coprime. Denote by $\clubsuit_{A,v}$ the condition for a sheaf $\mathscr{E}$ on a fiber of $\pi^{-1}(b)$ to be Gieseker stable with respect to A and satisfy*

$$\mathrm{rk}\,\mathscr{E} = r, \quad c_1(\mathscr{E}) = B|_{\pi^{-1}(b)}, \quad \textit{and} \quad \chi(\mathscr{E}) = r + s.$$

Denote by $M_{A,v}$ the open subfunctor of $\mathbf{M}$ corresponding to $\clubsuit_{A,v}$. This functor is representable by a projective $\mathbf{P}^1$–scheme $M = M_{A,v}$.

Proof of Proposition 3.6. Since r and χ are coprime, if $\mathscr{E}$ is Gieseker semistable, then it is Gieseker stable. Therefore, $\mathbf{M}_{H,v}$ agrees with with the relative moduli functor of Gieseker semistable sheaves on Z with Mukai vector v. Simpson [Sim94, Section 1] proved that this functor is universally corepresented by a proper and separated $\mathbf{P}^1$–scheme $M_{H,v}$. Since r and χ are coprime, it follows from [HL10, Corollary 4.6.7] that $M_{H,v}$ carries a universal sheaf and thus represents $\mathbf{M}_{H,v}$. □

3.3 Sheaves on K3 surfaces

Definition 3.7. Let Σ be a K3 surface. The **Mukai lattice** of Σ is $\tilde{H}(Z) = \mathbf{Z} \oplus H^2(\Sigma, \mathbf{Z}) \oplus \mathbf{Z}$ with the quadratic form given by

$$(r, B, s)^2 = B^2 - 2rs.$$

Let $\mathscr{E}$ be a coherent sheaf on Σ. The **Mukai vector** is defined by

$$v(\mathscr{E}) \coloneqq (\mathrm{rk}(\mathscr{E}), c_1(\mathscr{E}), \chi(\mathscr{E}) - \mathrm{rk}(\mathscr{E})) \in \mathbf{N}_0 \oplus H^{1,1}(\Sigma, \mathbf{Z}) \oplus \mathbf{Z} \subset \tilde{H}(\Sigma).$$

Proposition 3.8 ([HL10, Corollary 6.1.5]). *For every coherent sheaf $\mathscr{E}$ on a K3 surface*

$$\dim \mathrm{Ext}^1(\mathscr{E}, \mathscr{E}) = 2 \dim H^0(\mathscr{E}nd(\mathscr{E})) - v(\mathscr{E})^2.$$

Theorem 3.9 ([Huy15, Theorem 10.2.7]). *Let (Σ, A) be a polarized smooth K3 surface. For every Mukai vector $v = (r, B, s) \in \mathbf{N} \oplus H^{1,1}(\Sigma, \mathbf{Z}) \oplus \mathbf{Z}$ with $v^2 \geqslant -2$ and r and s coprime there exists a Gieseker stable sheaf $\mathscr{E}$ over Σ with*

$$v(\mathscr{E}) = v.$$

Theorem 3.10 (Mukai [Muk87, Proposition 3.3 and Corollaries 3.5 and 3.6] and Thomas [Thooo, Theorem 4.5]). *Let (Σ, A) be a polarised K3 surface with at worst RDP singularities. If $\mathscr{E}$ is a Gieseker stable sheaf with*

$$v(\mathscr{E})^2 = -2,$$

then it is locally free and any other Gieseker semistable sheaf with the same Mukai vector is isomorphic to $\mathscr{E}$. In particular, the moduli space of Gieseker stable sheaves with Mukai vector $v(\mathscr{E})$ is a reduced point.

Proposition 3.11. *Let (Σ, A) be a polarized smooth K3 surface and let $\mathscr{E}$ be a Gieseker stable sheaf on Σ. If $\operatorname{rk}\mathscr{E}$ and the divisibility of $c_1(\mathscr{E})$ are coprime and for every non-zero $x \in H^{1,1}(\Sigma, \mathbf{Z})$ perpendicular to $c_1(A)$*

$$x^2 < -\frac{1}{4}(\operatorname{rk}\mathscr{E})^2\big(2(\operatorname{rk}\mathscr{E})^2 + v(\mathscr{E})^2\big), \tag{3.12}$$

then $\mathscr{E}$ is μ-stable.

Proof. Since $\mathscr{E}$ is Gieseker stable, it is μ-semistable. Suppose $\mathscr{F}$ were a destabilizing sheaf, that is, a torsion-free subsheaf $\mathscr{F} \subset \mathscr{E}$ with $0 < \operatorname{rk}(\mathscr{F}) < \operatorname{rk}\mathscr{E}$ and $\mu(\mathscr{F}) = \mu(\mathscr{E})$. Set

$$x \coloneqq \operatorname{rk}\mathscr{E} \cdot c_1(\mathscr{F}) - \operatorname{rk}\mathscr{F} \cdot c_1(\mathscr{E}).$$

This expression is non-zero because $\operatorname{rk}\mathscr{E}$ and the divisibility of $c_1(\mathscr{E})$ are coprime. Since

$$x \cdot c_1(A) = \operatorname{rk}\mathscr{E} \cdot \operatorname{rk}\mathscr{F} \cdot (\mu(\mathscr{F}) - \mu(\mathscr{E})) = 0,$$

x satisfies (3.12).

Define the discriminant of $\mathscr{E}$ by

$$\Delta(\mathscr{E}) \coloneqq 2\operatorname{rk}\mathscr{E} \cdot c_2(\mathscr{E}) - (\operatorname{rk}\mathscr{E} - 1)c_1(\mathscr{E})^2.$$

According to [HL10, Theorem 4.C.3]

$$-\frac{1}{4}(\operatorname{rk}\mathscr{E})^2\Delta(\mathscr{E}) \leqslant x^2.$$

By Hirzebruch–Riemann–Roch $\Delta(\mathscr{E})$ can be rewritten as

$$\Delta(\mathscr{E}) = v(\mathscr{E})^2 + 2(\operatorname{rk}\mathscr{E})^2$$

leading to a contradiction x satisfying (3.12). □

3.4 Proof of Theorem 3.1

Choose $\varepsilon \ll 1$ and construct $(Z_\pm, \pi_\pm, \boldsymbol{\omega}_\pm; \mathfrak{r})$ accordingly via Proposition 2.11. By Remark 2.6 it can be arranged that $\pi_\pm$ has only finitely many singular fibres and the singular fibres have only ordinary double point singularities.

By slight abuse of notation identify $\mathrm{Pic}(W_\pm)$ and $\pi_\pm^* \mathrm{Pic}(W_\pm) \subset \mathrm{Pic}(Z_\pm)$. In particular, identify $H_\pm$ with the corresponding $\pi_\pm$–ample line bundle on $Z_\pm$. Set $v_\pm \coloneqq (r, h_\pm i_\pm(B), s)$. By Proposition 3.6, the open subfunctor of $\mathbf{M}$ corresponding to $\clubsuit_{H_\pm, v_\pm}$ is representable by a projective $\mathbf{P}^1$–scheme $\varpi_\pm \colon M_\pm \to \mathbf{P}^1$. Choose a universal sheaf $\mathscr{U}_\pm$ over $Z_\pm \times_{P^1} M_\pm$. It follows from Theorem 3.9 and Theorem 3.10 that $M_\pm = \mathbf{P}^1$ and $\varpi_\pm = \mathrm{id}_{\mathbf{P}^1}$. Therefore, $\mathscr{U}_\pm$ is a sheaf over $Z_\pm$. By Theorem 3.10, the restriction of $\mathscr{U}_\pm$ to every fiber of $\pi_\pm$ is locally free; hence, by [Sim94, Lemma 1.27], $\mathscr{U}_\pm$ is locally free.

It remains to show that Theorem 3.2 applies with $\mathscr{E}_\pm = \mathscr{U}_\pm$. Hypothesis (1) holds by construction. By Proposition 3.11, $\mathscr{U}_\pm|_{\pi_\pm^*(\infty)}$ is μ–stable with respect to $H_\pm$. Since $\mathbf{R}h_\pm(H_\pm)$ and $\mathbf{R}[\omega_I^\pm]$ have distance at most ε and $\varepsilon \ll 1$ it also is μ–stable with respect to $\omega_I^\pm$. By construction for every $b \in \mathbf{P}^1$, $H^1(\pi_\pm^{-1}(b), \mathscr{E}\mathrm{nd}_0(\mathscr{U}_\pm)) = 0$; hence, by Grothendieck's spectral sequence, $H^1(Z_\pm, \mathscr{E}\mathrm{nd}_0(\mathscr{U}_\pm)) = 0$. This shows that hypothesis (3) holds as well. □

4 How to find examples?

4.1 Constructing orthogonal pushouts

Let $N_\pm$ and N_0 be non-degenerate lattices and let $i_\pm \colon N_0 \hookrightarrow N_\pm$ be primitive embeddings. If there exists a pushout of i_+ and i_-, then it is unique up to isomorphism. The following procedure constructs the orthogonal pushout if it does exist.

Choose bases $H_1^\pm, \ldots, H_{\rho_\pm}^\pm$ of $N_\pm$, $B_1, \ldots, B_{\rho_0}$ of N_0, and $B_{\rho_0+1}^\pm, \ldots, B_{\rho_\pm}^\pm$ of $N_0^\perp \subset N_\pm$. The vectors $i_\pm(B_1), \ldots, i_\pm(B_{\rho_0}), B_{\rho_0+1}^\pm, \ldots, B_{\rho_\pm}^\pm$ span a sublattice of $N^\pm$. Identifying this lattice with $\mathbf{Z}^{\rho_\pm}$ the quadratic form is encoded in an integral matrix of the form

$$\begin{pmatrix} q_0 & 0 \\ 0 & q_\pm \end{pmatrix}$$

with $q_0 \in \mathbf{Z}^{\rho_0 \times \rho_0}$ and $q_\pm \in \mathbf{Z}^{(\rho_\pm - \rho_0) \times (\rho_\pm - \rho_0)}$. Define $T_\pm \in \mathbf{Z}^{\rho_\pm \times \rho_\pm}$ by

$$\begin{pmatrix} i_\pm(B_1) & \cdots & i_\pm(B_{\rho_0}) & B_{\rho_0+1}^\pm & \cdots & B_{\rho_\pm}^\pm \end{pmatrix} = \begin{pmatrix} H_1^\pm & \cdots & H_{\rho_\pm}^\pm \end{pmatrix} T_\pm.$$

The columns of the *rational* matrix $T_\pm^{-1} \in \mathbf{Q}^{\rho_\pm \times \rho_\pm}$ represent the basis vectors of $N_\pm$. This gives a presentation of $N_\pm$ as an overlattice of $\mathbf{Z}^{\rho_\pm}$ with the above quadratic form.

Define inclusions $j_\pm \colon \mathbf{Q}^{\rho_\pm} \hookrightarrow \mathbf{Q}^r$ by

$$\begin{aligned} j_+(x_1, \ldots, x_{\rho_+}) &\coloneqq (x_1, \ldots, x_{\rho_+}, 0, \ldots, 0) \quad \text{and} \\ j_-(x_1, \ldots, x_{\rho_-}) &\coloneqq (x_1, \ldots, x_{\rho_0}, 0, \ldots, 0, x_{\rho_0+1}, \ldots, x_{\rho_-}). \end{aligned}$$

Denote by N the subgroup of $\mathbf{Q}^r$ generated by the images the columns of T_+^{-1} and T_-^{-1} under j_+ and j_-. The matrix

$$\begin{pmatrix} q_0 & 0 & 0 \\ 0 & q_+ & 0 \\ 0 & 0 & q_- \end{pmatrix}$$

defines a *rational* quadratic form on N. If this quadratic form takes values in the integers, then N is a lattice and together with the primitive embeddings $j_\pm \colon N_\pm \hookrightarrow N$ forms a orthogonal pushout of i_+ and i_-. Whether the quadratic form is integral or not is easily checked by computing the products between the images of the columns of T_+^{-1} under j_+ and the images of the columns of T_-^{-1} under j_-.

4.2 Finding input for Theorem 3.1

The deformation types of Fano 3–folds have been classified by Mori and Mukai [MM81]. Appendix B lists all deformation types of Fano 3–folds W with $\rho := \operatorname{rk}\operatorname{Pic}(W) \in \{2,3\}$. The entries of this list are labeled by $\#_n^\rho$. For each $\#_n^\rho$ the Picard lattice $\operatorname{Pic}(W)$ is given in terms of a concrete basis $H_1, \dots, H_\rho$ of $\operatorname{Pic}(W)$ such that the interior of the cone

$$\mathbf{R}_+ H_1 + \cdots + \mathbf{R}_+ H_\rho$$

is contained in the ample cone of W. All of the entries in Appendix B except to listed in Appendix C have very ample anticanonical bundle.

Equipped with this data one can try to find examples of the input required for Theorem 3.1 by executing the following steps:

0. Let $r \in \{2, 3, \dots\}$. Pick two entries in Appendix B except those listed in Appendix C and denote the corresponding lattices by $N_\pm$.

1. Try to find a pair vectors $B_\pm \in N_\pm$ such that:

   (a) $B_+^2 = B_-^2$,

   (b) $s := \frac{B_+^2 + 2}{2r} \in \mathbf{Z}$,

   (c) the r and the divisibility of B are coprime, and r and s are coprime.

2. Suppose a suitable pair $B_\pm$ has been found. Try to find $H_\pm \in B_\pm^\perp$ whose representation in terms of $H_1^\pm, \dots, H_{\rho_\pm}^\pm$ has positive coefficients.

3. Suppose suitable $H_\pm$ have been found. Compute the maximal integer represented by the quadratic form on $H_\pm^\perp$. Check that this value is less than $-\frac{1}{2}r^2(r^2-1)$.

4. Suppose that the check in the last step has been passed. Denote by N_0 the rank one lattice with quadratic form (B_+^2) and by $i_\pm \colon N_0 \to N_\pm$ the primitive embedding defined by $B_\pm$. Check that the orthogonal pushout of i_+ and i_- exists.

If this final check also passes, then the data found in the process gives the input required for Theorem 3.1.

Steps (1) and (2) can be carried out by a brute-force search. Step (3) is a trivial task if $\rho_\pm = 2$. If $\rho_\pm = 3$, then it can be efficiently carried out as follows. Choose some basis of $H_\pm^\perp$ and write the quadratic form as $-ax^2 - bxy - cy^2$. Using Gauss–Lagrange's algorithm one may assume that the quadratic form is reduced; that is: $1 \leqslant a \leqslant c$, $|b| \leqslant a$, and if $a = c$, then $b \geqslant 0$. The maximal integer represented by the quadratic form is then $-a$. In general, (3) can be carried out efficiently using the algorithm from [ER01]. Finally, step (4) can be carried out using the procedure from Section 4.1. All of this is easily implemented in a computer program. A concrete implementation in SAGE/PYTHON is available at `https://walpu.ski/Research/ArithmeticG2InstantonsTCS.zip`.

5 Examples found by brute-force search

The brute-force search with scope for $B_\pm$ and $H_\pm$ restricted to $\{-20, \dots, 20\}^\rho \subset \mathbf{Z}^\rho$ yields 299 instances of the input required for Theorem 3.1. Table ?? gives more detailed statistics regarding these. In this table r refers to the rank of the bundle and ${}^{\rho_1}_{n_1}\#^{\rho_2}_{n_2}$ means that the matching pair of building blocks come from the Fano 3–folds listed as $\#^{\rho_1}_{n_1}$ and $\#^{\rho_2}_{n_2}$ in Appendix B. Repeated occurrences of ${}^{\rho_1}_{n_1}\#^{\rho_2}_{n_2}$ indicate multiple possible choices for $B_\pm$. The first entry in Table ?? (${}^{2}_{13}\#^{2}_{14}$) recovers the example from [Wal16]. The other entries are all new. in particular, this yields the first examples of irreducible, unobstructed G_2–instantons on $\mathbf{PU}(r)$–bundles with $r \neq 2$. It should be pointed the rank 7 examples are distinct from the Levi-Civita connection on the underlying G_2–manifolds. To see this observe that on a G_2–manifold (Y, ϕ) the 3–form δ defines an element of $\Omega^1(TY, \mathrm{End}(TY))$ which is a non-trivial infinitesimal deformation of the Levi-Civita connection. Therefore, the latter cannot be rigid/unobstructed.

A How to compute Picard lattices?

The data in Appendix B has been computed using the following well-known results.

Proposition A.1 (Divisors). *Let X be a smooth complex 4–fold, L a line bundle over X, and W a divisor in X. Denote by $i\colon W \to X$ the inclusion. The anticanonical bundle of W is given by*

$$-K_W = -i^*(K_X + L).$$

The map $i^*\colon \mathrm{Pic}(X) \to \mathrm{Pic}(W)$ is an isomorphism of abelian groups and for every $A, B \in \mathrm{Pic}(X)$

$$i^*A \cdot i^*B \cdot (-K_W) = A \cdot B \cdot L \cdot (-K_X - L).$$

If $A \in \mathrm{Pic}(X)$ is nef, then so is i^*A.

r	B^2	matching pairs	
2	-30	${}^{2}_{13}\#^{2}_{14}$, ${}^{2}_{13}\#^{3}_{25}$, ${}^{2}_{13}\#^{3}_{25}$	3
	-110	${}^{2}_{16}\#^{3}_{21}$, ${}^{2}_{16}\#^{3}_{24}$, ${}^{3}_{10}\#^{3}_{21}$, ${}^{3}_{10}\#^{3}_{24}$, ${}^{3}_{10}\#^{3}_{21}$, ${}^{3}_{10}\#^{3}_{24}$, ${}^{3}_{16}\#^{3}_{21}$, ${}^{3}_{16}\#^{3}_{24}$, ${}^{3}_{16}\#^{3}_{21}$, ${}^{3}_{16}\#^{3}_{24}$	10
	-270	${}^{2}_{13}\#^{2}_{14}$, ${}^{2}_{13}\#^{3}_{25}$, ${}^{2}_{13}\#^{3}_{25}$	3
	-510	${}^{3}_{8}\#^{3}_{16}$, ${}^{3}_{8}\#^{3}_{16}$, ${}^{3}_{15}\#^{3}_{16}$	3
	-750	${}^{2}_{13}\#^{2}_{14}$, ${}^{2}_{13}\#^{3}_{25}$, ${}^{2}_{13}\#^{3}_{25}$	3
	-990	${}^{3}_{16}\#^{3}_{21}$	1
	-2750	${}^{3}_{16}\#^{3}_{21}$	1
3	-224	${}^{2}_{7}\#^{3}_{3}$, ${}^{2}_{7}\#^{3}_{3}$, ${}^{2}_{7}\#^{3}_{5}$, ${}^{2}_{7}\#^{3}_{11}$, ${}^{2}_{7}\#^{3}_{17}$, ${}^{2}_{7}\#^{3}_{17}$, ${}^{2}_{7}\#^{3}_{30}$, ${}^{2}_{29}\#^{3}_{3}$, ${}^{2}_{29}\#^{3}_{3}$, ${}^{2}_{29}\#^{3}_{5}$, ${}^{2}_{29}\#^{3}_{11}$, ${}^{2}_{29}\#^{3}_{17}$, ${}^{2}_{29}\#^{3}_{17}$, ${}^{2}_{29}\#^{3}_{30}$	14
	-440	${}^{2}_{16}\#^{3}_{21}$, ${}^{2}_{16}\#^{3}_{24}$, ${}^{3}_{10}\#^{3}_{21}$, ${}^{3}_{10}\#^{3}_{24}$, ${}^{3}_{10}\#^{3}_{21}$, ${}^{3}_{10}\#^{3}_{24}$, ${}^{3}_{16}\#^{3}_{21}$, ${}^{3}_{16}\#^{3}_{24}$, ${}^{3}_{16}\#^{3}_{21}$, ${}^{3}_{16}\#^{3}_{24}$	10
	-896	${}^{2}_{7}\#^{3}_{3}$, ${}^{2}_{7}\#^{3}_{3}$, ${}^{2}_{7}\#^{3}_{5}$, ${}^{2}_{7}\#^{3}_{11}$, ${}^{2}_{7}\#^{3}_{17}$, ${}^{2}_{7}\#^{3}_{17}$, ${}^{2}_{7}\#^{3}_{30}$, ${}^{2}_{29}\#^{3}_{3}$, ${}^{2}_{29}\#^{3}_{3}$, ${}^{2}_{29}\#^{3}_{5}$, ${}^{2}_{29}\#^{3}_{11}$, ${}^{2}_{29}\#^{3}_{17}$, ${}^{2}_{29}\#^{3}_{17}$, ${}^{2}_{29}\#^{3}_{30}$	14
	-1088	${}^{2}_{9}\#^{3}_{28}$, ${}^{2}_{27}\#^{3}_{28}$, ${}^{3}_{8}\#^{3}_{28}$	3
	-1760	${}^{3}_{16}\#^{3}_{21}$	1
	-2750	${}^{3}_{16}\#^{3}_{21}$	1
4	-90	${}^{3}_{10}\#^{3}_{22}$, ${}^{3}_{10}\#^{3}_{22}$, ${}^{3}_{10}\#^{3}_{26}$, ${}^{3}_{10}\#^{3}_{22}$, ${}^{3}_{10}\#^{3}_{22}$, ${}^{3}_{10}\#^{3}_{26}$	6
5	-112	${}^{3}_{3}\#^{3}_{5}$, ${}^{3}_{5}\#^{3}_{5}$, ${}^{3}_{5}\#^{3}_{5}$, ${}^{3}_{5}\#^{3}_{5}$, ${}^{3}_{5}\#^{3}_{11}$, ${}^{3}_{5}\#^{3}_{17}$, ${}^{3}_{5}\#^{3}_{29}$, ${}^{3}_{5}\#^{3}_{29}$	8
	-272	${}^{3}_{2}\#^{3}_{8}$, ${}^{3}_{2}\#^{3}_{15}$, ${}^{3}_{2}\#^{3}_{8}$, ${}^{3}_{2}\#^{3}_{15}$, ${}^{3}_{8}\#^{3}_{28}$, ${}^{3}_{8}\#^{3}_{28}$, ${}^{3}_{15}\#^{3}_{28}$, ${}^{3}_{15}\#^{3}_{28}$	8
6	-2750	${}^{3}_{16}\#^{3}_{21}$	1
7	-16	${}^{3}_{1}\#^{3}_{1}$, ${}^{3}_{1}\#^{3}_{3}$, ${}^{3}_{1}\#^{3}_{7}$, ${}^{3}_{1}\#^{3}_{13}$, ${}^{3}_{1}\#^{3}_{17}$, ${}^{3}_{1}\#^{3}_{27}$, ${}^{3}_{1}\#^{3}_{28}$, ${}^{3}_{1}\#^{3}_{31}$, ${}^{3}_{3}\#^{3}_{3}$, ${}^{3}_{3}\#^{3}_{7}$, ${}^{3}_{3}\#^{3}_{13}$, ${}^{3}_{3}\#^{3}_{17}$, ${}^{3}_{3}\#^{3}_{27}$, ${}^{3}_{3}\#^{3}_{28}$, ${}^{3}_{3}\#^{3}_{31}$, ${}^{3}_{7}\#^{3}_{7}$, ${}^{3}_{7}\#^{3}_{13}$, ${}^{3}_{7}\#^{3}_{17}$, ${}^{3}_{7}\#^{3}_{27}$, ${}^{3}_{7}\#^{3}_{28}$, ${}^{3}_{7}\#^{3}_{31}$, ${}^{3}_{13}\#^{3}_{13}$, ${}^{3}_{13}\#^{3}_{17}$, ${}^{3}_{13}\#^{3}_{27}$, ${}^{3}_{13}\#^{3}_{28}$, ${}^{3}_{13}\#^{3}_{31}$, ${}^{3}_{17}\#^{3}_{17}$, ${}^{3}_{17}\#^{3}_{27}$, ${}^{3}_{17}\#^{3}_{28}$, ${}^{3}_{17}\#^{3}_{31}$, ${}^{3}_{27}\#^{3}_{27}$, ${}^{3}_{27}\#^{3}_{28}$, ${}^{3}_{27}\#^{3}_{31}$, ${}^{3}_{28}\#^{3}_{28}$, ${}^{3}_{28}\#^{3}_{31}$, ${}^{3}_{31}\#^{3}_{31}$	36
	-324	${}^{3}_{1}\#^{3}_{1}$, ${}^{3}_{1}\#^{3}_{3}$, ${}^{3}_{1}\#^{3}_{7}$, ${}^{3}_{1}\#^{3}_{12}$, ${}^{3}_{1}\#^{3}_{13}$, ${}^{3}_{1}\#^{3}_{17}$, ${}^{3}_{1}\#^{3}_{27}$, ${}^{3}_{1}\#^{3}_{31}$, ${}^{3}_{3}\#^{3}_{3}$, ${}^{3}_{3}\#^{3}_{7}$, ${}^{3}_{3}\#^{3}_{12}$, ${}^{3}_{3}\#^{3}_{13}$, ${}^{3}_{3}\#^{3}_{17}$, ${}^{3}_{3}\#^{3}_{27}$, ${}^{3}_{3}\#^{3}_{31}$, ${}^{3}_{7}\#^{3}_{7}$, ${}^{3}_{7}\#^{3}_{12}$, ${}^{3}_{7}\#^{3}_{13}$, ${}^{3}_{7}\#^{3}_{17}$, ${}^{3}_{7}\#^{3}_{27}$, ${}^{3}_{7}\#^{3}_{31}$, ${}^{3}_{12}\#^{3}_{12}$, ${}^{3}_{12}\#^{3}_{13}$, ${}^{3}_{12}\#^{3}_{17}$, ${}^{3}_{12}\#^{3}_{27}$, ${}^{3}_{12}\#^{3}_{31}$, ${}^{3}_{13}\#^{3}_{13}$, ${}^{3}_{13}\#^{3}_{17}$, ${}^{3}_{13}\#^{3}_{27}$, ${}^{3}_{13}\#^{3}_{31}$, ${}^{3}_{17}\#^{3}_{17}$, ${}^{3}_{17}\#^{3}_{27}$, ${}^{3}_{17}\#^{3}_{31}$, ${}^{3}_{27}\#^{3}_{27}$, ${}^{3}_{27}\#^{3}_{31}$, ${}^{3}_{31}\#^{3}_{31}$	36
	-576	${}^{3}_{1}\#^{3}_{1}$, ${}^{3}_{1}\#^{3}_{3}$, ${}^{3}_{1}\#^{3}_{7}$, ${}^{3}_{1}\#^{3}_{12}$, ${}^{3}_{1}\#^{3}_{13}$, ${}^{3}_{1}\#^{3}_{17}$, ${}^{3}_{1}\#^{3}_{27}$, ${}^{3}_{1}\#^{3}_{28}$, ${}^{3}_{1}\#^{3}_{31}$, ${}^{3}_{3}\#^{3}_{3}$, ${}^{3}_{3}\#^{3}_{7}$, ${}^{3}_{3}\#^{3}_{12}$, ${}^{3}_{3}\#^{3}_{13}$, ${}^{3}_{3}\#^{3}_{17}$, ${}^{3}_{3}\#^{3}_{27}$, ${}^{3}_{3}\#^{3}_{28}$, ${}^{3}_{3}\#^{3}_{31}$, ${}^{3}_{7}\#^{3}_{7}$, ${}^{3}_{7}\#^{3}_{12}$, ${}^{3}_{7}\#^{3}_{13}$, ${}^{3}_{7}\#^{3}_{17}$, ${}^{3}_{7}\#^{3}_{27}$, ${}^{3}_{7}\#^{3}_{28}$, ${}^{3}_{7}\#^{3}_{31}$, ${}^{3}_{12}\#^{3}_{12}$, ${}^{3}_{12}\#^{3}_{13}$, ${}^{3}_{12}\#^{3}_{17}$, ${}^{3}_{12}\#^{3}_{27}$, ${}^{3}_{12}\#^{3}_{28}$, ${}^{3}_{12}\#^{3}_{31}$, ${}^{3}_{13}\#^{3}_{13}$, ${}^{3}_{13}\#^{3}_{17}$, ${}^{3}_{13}\#^{3}_{27}$, ${}^{3}_{13}\#^{3}_{28}$, ${}^{3}_{13}\#^{3}_{31}$, ${}^{3}_{17}\#^{3}_{17}$, ${}^{3}_{17}\#^{3}_{27}$, ${}^{3}_{17}\#^{3}_{28}$, ${}^{3}_{17}\#^{3}_{31}$, ${}^{3}_{27}\#^{3}_{27}$, ${}^{3}_{27}\#^{3}_{28}$, ${}^{3}_{27}\#^{3}_{31}$, ${}^{3}_{28}\#^{3}_{28}$, ${}^{3}_{28}\#^{3}_{31}$, ${}^{3}_{31}\#^{3}_{31}$	45
	-1024	${}^{3}_{1}\#^{3}_{1}$, ${}^{3}_{1}\#^{3}_{3}$, ${}^{3}_{1}\#^{3}_{7}$, ${}^{3}_{1}\#^{3}_{13}$, ${}^{3}_{1}\#^{3}_{17}$, ${}^{3}_{1}\#^{3}_{27}$, ${}^{3}_{1}\#^{3}_{28}$, ${}^{3}_{1}\#^{3}_{31}$, ${}^{3}_{3}\#^{3}_{3}$, ${}^{3}_{3}\#^{3}_{7}$, ${}^{3}_{3}\#^{3}_{13}$, ${}^{3}_{3}\#^{3}_{17}$, ${}^{3}_{3}\#^{3}_{27}$, ${}^{3}_{3}\#^{3}_{28}$, ${}^{3}_{3}\#^{3}_{31}$, ${}^{3}_{7}\#^{3}_{7}$, ${}^{3}_{7}\#^{3}_{13}$, ${}^{3}_{7}\#^{3}_{17}$, ${}^{3}_{7}\#^{3}_{27}$, ${}^{3}_{7}\#^{3}_{28}$, ${}^{3}_{7}\#^{3}_{31}$, ${}^{3}_{13}\#^{3}_{13}$, ${}^{3}_{13}\#^{3}_{17}$, ${}^{3}_{13}\#^{3}_{27}$, ${}^{3}_{13}\#^{3}_{28}$, ${}^{3}_{13}\#^{3}_{31}$, ${}^{3}_{17}\#^{3}_{17}$, ${}^{3}_{17}\#^{3}_{27}$, ${}^{3}_{17}\#^{3}_{28}$, ${}^{3}_{17}\#^{3}_{31}$, ${}^{3}_{27}\#^{3}_{27}$, ${}^{3}_{27}\#^{3}_{28}$, ${}^{3}_{27}\#^{3}_{31}$, ${}^{3}_{28}\#^{3}_{28}$, ${}^{3}_{28}\#^{3}_{31}$, ${}^{3}_{31}\#^{3}_{31}$	36
	-1444	${}^{3}_{1}\#^{3}_{1}$, ${}^{3}_{1}\#^{3}_{3}$, ${}^{3}_{1}\#^{3}_{7}$, ${}^{3}_{1}\#^{3}_{13}$, ${}^{3}_{1}\#^{3}_{17}$, ${}^{3}_{1}\#^{3}_{27}$, ${}^{3}_{1}\#^{3}_{31}$, ${}^{3}_{3}\#^{3}_{3}$, ${}^{3}_{3}\#^{3}_{7}$, ${}^{3}_{3}\#^{3}_{13}$, ${}^{3}_{3}\#^{3}_{17}$, ${}^{3}_{3}\#^{3}_{27}$, ${}^{3}_{3}\#^{3}_{31}$, ${}^{3}_{7}\#^{3}_{7}$, ${}^{3}_{7}\#^{3}_{13}$, ${}^{3}_{7}\#^{3}_{17}$, ${}^{3}_{7}\#^{3}_{27}$, ${}^{3}_{7}\#^{3}_{31}$, ${}^{3}_{13}\#^{3}_{13}$, ${}^{3}_{13}\#^{3}_{17}$, ${}^{3}_{13}\#^{3}_{27}$, ${}^{3}_{13}\#^{3}_{31}$, ${}^{3}_{17}\#^{3}_{17}$, ${}^{3}_{17}\#^{3}_{27}$, ${}^{3}_{17}\#^{3}_{31}$, ${}^{3}_{27}\#^{3}_{27}$, ${}^{3}_{27}\#^{3}_{31}$, ${}^{3}_{31}\#^{3}_{31}$	28
8	-306	${}^{3}_{14}\#^{3}_{15}$, ${}^{3}_{15}\#^{3}_{22}$, ${}^{3}_{15}\#^{3}_{26}$	3
13	-990	${}^{3}_{16}\#^{3}_{21}$	1
	-1224	${}^{3}_{14}\#^{3}_{15}$	1
17	-36	${}^{3}_{3}\#^{3}_{3}$, ${}^{3}_{3}\#^{3}_{12}$, ${}^{3}_{3}\#^{3}_{17}$, ${}^{3}_{3}\#^{3}_{31}$, ${}^{3}_{12}\#^{3}_{12}$, ${}^{3}_{12}\#^{3}_{17}$, ${}^{3}_{12}\#^{3}_{31}$, ${}^{3}_{17}\#^{3}_{17}$, ${}^{3}_{17}\#^{3}_{31}$, ${}^{3}_{31}\#^{3}_{31}$	10
	-784	${}^{3}_{3}\#^{3}_{3}$, ${}^{3}_{3}\#^{3}_{17}$, ${}^{3}_{3}\#^{3}_{31}$, ${}^{3}_{17}\#^{3}_{17}$, ${}^{3}_{17}\#^{3}_{31}$, ${}^{3}_{31}\#^{3}_{31}$	6
	-1600	${}^{3}_{3}\#^{3}_{3}$, ${}^{3}_{3}\#^{3}_{17}$, ${}^{3}_{3}\#^{3}_{31}$, ${}^{3}_{17}\#^{3}_{17}$, ${}^{3}_{17}\#^{3}_{31}$, ${}^{3}_{31}\#^{3}_{31}$	6
19	-306	${}^{3}_{14}\#^{3}_{15}$	1

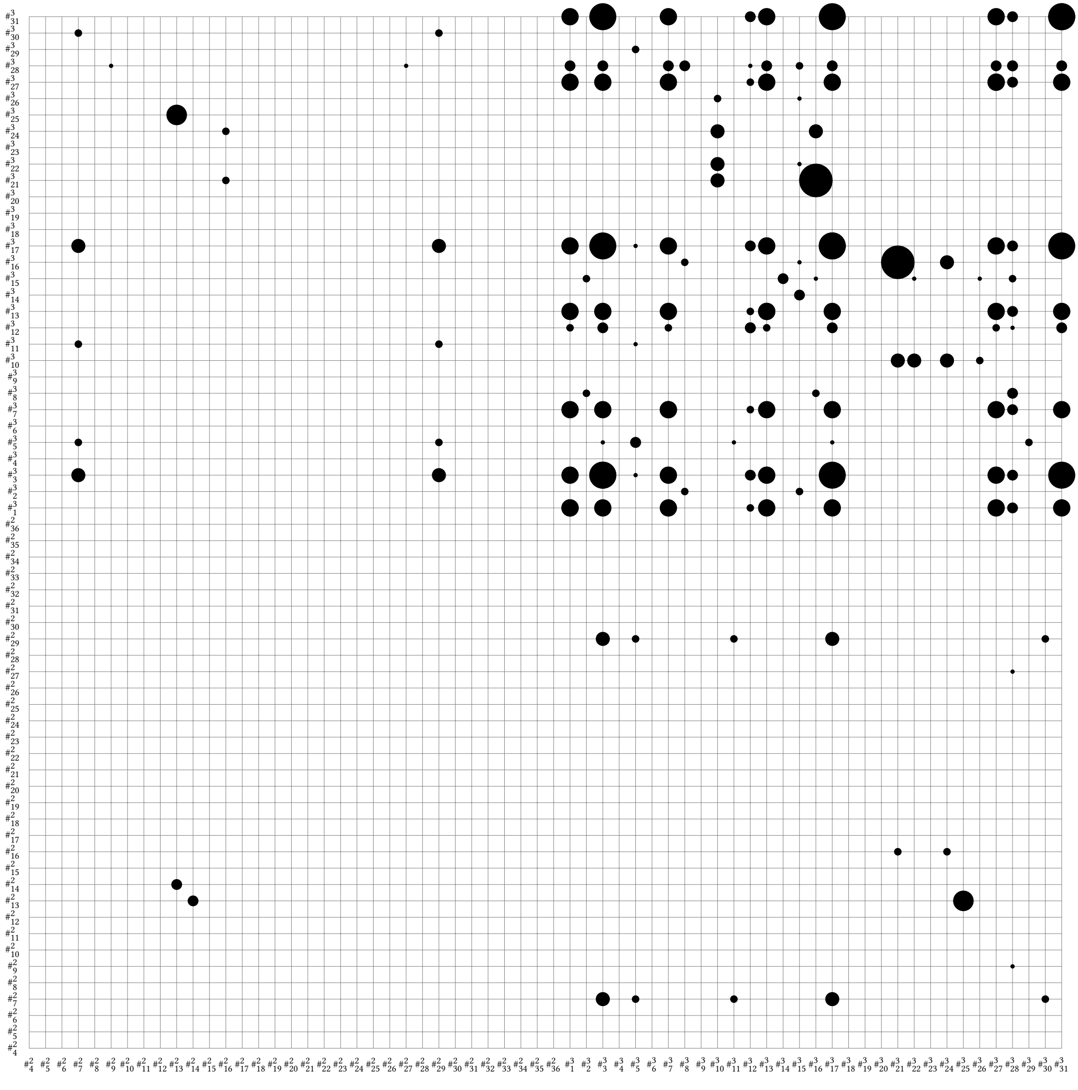

Figure 1

Proposition A.2 (Branched double covers). *Let W be a smooth complex 3–fold, let L be a line bundle over W, and let $\pi\colon \tilde{W} \to W$ be a double cover branched over a divisor in $|2L|$. The anticanonical bundle of $\tilde{W}$ is given by*

$$-K_{\tilde{W}} = -\pi^*(K_W + L).$$

The map $\pi\colon \operatorname{Pic}(W) \to \operatorname{Pic}(\tilde{W})$ is an isomorphism of abelian groups and for every $A, B \in \operatorname{Pic}(W)$

$$\pi^*A \cdot \pi^*B \cdot -K_{\tilde{W}} = 2A \cdot B \cdot (-K_W - L).$$

If $A \in \operatorname{Pic}(W)$ is nef, then so is π^*A.

Definition A.3. Let X be a complex manifold and let L be a line bundle over X, and let $Z \subset X$ be a smooth complex submanifold. Denote by $\mathscr{I}_Z$ the ideal sheaf of Z. The submanifold Z is said to be **cut-out by sections of** L if $L \otimes \mathscr{I}_Z$ is globally generated; that is: for every $x \in Z$ there is a neighborhood U of x such that every section of L defined over U and which vanishes on Z extends to a global section of L vanishing on Z.

Proposition A.4 (Blow-up in a point [CN14, Lemma 4.5; EH16, Section 13.6]). *Let W be a smooth complex 3–fold and let $\pi\colon \tilde{W} \to W$ be the blow-up of W in a point x. Denote by E the exceptional divisor. The anticanonical bundle of $\tilde{W}$ is given by*

$$-K_{\tilde{W}} = -\pi^*K_W - 2E.$$

As abelian groups $\operatorname{Pic}(\tilde{W}) = \pi^* \operatorname{Pic}(W) \oplus \langle E\rangle$ *and for every $A, B \in \operatorname{Pic}(W)$*

$$\pi^*A \cdot \pi^*B \cdot (-K_{\tilde{W}}) = A \cdot B \cdot (-K_W), \quad \pi^*A \cdot E \cdot (-K_{\tilde{W}}) = 0, \quad \text{and} \quad E \cdot E \cdot (-K_{\tilde{W}}) = -2.$$

If $A \in \operatorname{Pic}(W)$ is nef, then so is π^*A. If $\{x\}$ is cut-out be sections of L, then $\pi^*L - E$ is nef.

Proposition A.5 (Blow-up in a smooth curve [CN14, Lemma 4.5; EH16, Section 13.6]). *Let W be a smooth complex 3–fold and let $\pi\colon \tilde{W} \to W$ be the blow-up of W in a smooth curve C. Denote by E the exceptional divisor. The anticanonical bundle of $\tilde{W}$ is given by*

$$-K_{\tilde{W}} = -\pi^*K_W - E.$$

As abelian groups $\operatorname{Pic}(\tilde{W}) = \pi^* \operatorname{Pic}(W) \oplus \langle E\rangle$ *and for every $A, B \in \operatorname{Pic}(W)$*

$$\pi^*A \cdot \pi^*B \cdot (-K_{\tilde{W}}) = A \cdot B \cdot (-K_W), \quad \pi^*A \cdot E \cdot (-K_{\tilde{W}}) = \deg_A(C), \quad \text{and} \quad E \cdot E \cdot (-K_{\tilde{W}}) = -\chi(C).$$

If $A \in \operatorname{Pic}(W)$ is nef, then so is π^*A. If C is cut-out be sections of L, then $\pi^*L - E$ is nef.

Proposition A.6 ([GH94, p. 606; EH16, Theorem 9.6]). *Let X be a smooth n–fold and let E be a holomorphic vector bundle over X of rank r. Denote by $\pi\colon \mathbf{P}E \to X$ the $\mathbf{P}^{r-1}$–bundle associated with E and denote by $\mathscr{O}_{\mathbf{P}E}(1)$ the dual of the tautological line bundle over $\mathbf{P}E$. The anticanonical bundle of $\mathbf{P}E$ is given by*

$$-K_{\mathbf{P}E} = -\pi^*K_X + \det E + r\mathscr{O}_{\mathbf{P}E}(1).$$

$H^*(\mathbf{P}E)$ is a $H^*(X)$–algebra generated by $h = c_1(\mathscr{O}_{\mathbf{P}E}(1))$ subject to the relation

$$h^r + c_1(E)h^{r-1} + c_2(E)h^{r-2} + \cdots + c_r(E) = 0.$$

In particular, $\operatorname{Pic}(\mathbf{P}E) = \pi^* \operatorname{Pic}(X) \oplus \langle \mathscr{O}_{\mathbf{P}E}(1)\rangle$. *Moreover, if $A \in \operatorname{Pic}(X)$ is nef, then so is π^*A and if E^* is a direct sum of nef line bundles, then $\mathscr{O}_{\mathbf{P}E}(1)$ is nef.*

B Data for Fano 3–folds

The following list contains descriptions of a number of Fano 3–folds W together with generators $H_1, \ldots, H_r$ of $\mathrm{Pic}(W)$, the intersection form on $N = \mathrm{Pic}(W)$, and $-K_W$. These generators are chosen such that the cone $\mathbf{R}_+ H_1 + \cdots \mathbf{R}_+ H_r$ is contained in the nef cone $\overline{\mathrm{Amp}}(W)$ of W.[1] The entry $\#^r_n$ concerns the Fano 3–fold $W = W^r_n$ with $\mathrm{rk}\,\mathrm{Pic}(W) = r$ appearing as the nth entry of the corresponding table in [IP99, Chapter 12]. This data has been computed using the tools discussed in Appendix A. V_d for $d = 1, \ldots, 5$ refers to the del Pezzo Fano 3–fold of degree d that is a Fano 3–fold with $\mathrm{Pic}(V_d) = \langle -\frac{1}{2} K_{V_d} \rangle$ and $-K_d^3 = 8 \cdot d$.

	description	basis of Pic(W)	N	$-K_W$
$\#^2_1$	W is the blow-up of V_1 in an elliptic curve which is the intersection of two divisors in $\lvert -\frac{1}{2}K_{V_1}\rvert$.	$\pi^*(-\frac{1}{2}K_{V_1}), \pi^*(-\frac{1}{2}K_{V_1}) - E$	$\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
$\#^2_2$	W is a double cover of $\mathbf{P}^1 \times \mathbf{P}^2$ branched over a divisor of bidegree $(2, 4)$.	$\pi^*\mathscr{O}_{\mathbf{P}^1\times\mathbf{P}^2}(1, 0)$, $\pi^*\mathscr{O}_{\mathbf{P}^1\times\mathbf{P}^2}(0, 1)$	$\begin{pmatrix} 0 & 2 \\ 2 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
$\#^2_3$	W is the blow-up of V_2 in an elliptic curve which is an intersection of two divisors in $\lvert -\frac{1}{2}K_{V_2}\rvert$.	$\pi^*(-\frac{1}{2}K_{V_2}), \pi^*(-\frac{1}{2}K_{V_2}) - E$	$\begin{pmatrix} 4 & 2 \\ 2 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
$\#^2_4$	W is the blow-up of $\mathbf{P}^3$ in an intersection of two cubic hypersurfaces.	$\pi^*\mathscr{O}_{\mathbf{P}^3}(1), 3\pi^*\mathscr{O}_{\mathbf{P}^3}(1) - E$	$\begin{pmatrix} 4 & 3 \\ 3 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
$\#^2_5$	W is the blow up of $V_3 \subset \mathbf{P}^4$ along the intersection of two hyperplane divisors.	$\pi^*(-\frac{1}{2}K_{V_3}), \pi^*(-\frac{1}{2}K_{V_3}) - E$	$\begin{pmatrix} 6 & 3 \\ 3 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
$\#^2_6$	W is a divisor in $\mathbf{P}^2 \times \mathbf{P}^2$ of bidegree $(2, 2)$ or a double cover of W^2_{32} (a divisor in $\mathbf{P}^2 \times \mathbf{P}^2$ of bidegree $(1, 1)$) branched over an anticanonical divisor.	$i^*\mathscr{O}_{\mathbf{P}^2\times\mathbf{P}^2}(1, 0), i^*\mathscr{O}_{\mathbf{P}^2\times\mathbf{P}^2}(0, 1)$	$\begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

[1] This inclusion may be strict. Indeed, there are a few of instances where $-K_W$ is not contained in the former cone.

$\#^2_7$	W is the blow-up of a quadric hypersurface Q in $\mathbf{P}^4$ in the intersection of two quadrics.	$\pi^*(-\frac{1}{3}K_Q)^1$, $\pi^*(-\frac{1}{3}K_Q) - E$	$\begin{pmatrix} 6 & 4 \\ 4 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
$\#^2_8$	W is a double cover of W^2_{35} (V_7)) branched over a curve in $\lvert -K_{V_7}\rvert$ whose intersection with the exceptional divisor in V_7 is either smooth or reduced but not smooth.	the pull-backs of the generators of $\mathrm{Pic}(V_7)$ as stated below	$\begin{pmatrix} 2 & 4 \\ 4 & 4 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
$\#^2_9$	W is the blow-up of $\mathbf{P}^3$ along a curve of degree 7 and genus 5 which is an intersection of a family of cubic hypersurfaces.	$\pi^*\mathscr{O}_{\mathbf{P}^3}(1)$, $\pi^*\mathscr{O}_{\mathbf{P}^3}(3) - E$	$\begin{pmatrix} 4 & 5 \\ 5 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
$\#^2_{10}$	W is the blow-up of $V_4 \subset \mathbf{P}^5$ in an elliptic curve which is the intersection of two hyperplane sections.	$\pi^*(-\frac{1}{2}K_{V_4})$, $\pi^*(-\frac{1}{2}K_{V_4}) - E$	$\begin{pmatrix} 8 & 4 \\ 4 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
$\#^2_{11}$	W is the blow-up of $V_3 \subset \mathbf{P}^4$ along a line.	$\pi^*(-\frac{1}{2}K_{V_3})$, $\pi^*(-\frac{1}{2}K_{V_3}) - E$	$\begin{pmatrix} 6 & 5 \\ 5 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
$\#^2_{12}$	W is the blow-up of $\mathbf{P}^3$ along a curve of degree 6 and genus 3 which is an intersection of a family of cubic hypersurfaces.	$\pi^*\mathscr{O}_{\mathbf{P}^3}(1)$, $\pi^*\mathscr{O}_{\mathbf{P}^3}(3) - E$	$\begin{pmatrix} 4 & 6 \\ 6 & 4 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
$\#^2_{13}$	W is the blow-up of a quadric $Q \subset \mathbf{P}^4$ along a curve of degree 6 and genus 2.	$\pi^*(-\frac{1}{3}K_Q)$, $\pi^*(-\frac{2}{3}K_Q) - E$	$\begin{pmatrix} 6 & 6 \\ 6 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
$\#^2_{14}$	W is the blow-up of $V_5 \subset \mathbf{P}^9$ in an elliptic curve which is the intersection of two hyperplane sections.	$\pi^*(-\frac{1}{2}K_{V_5})$, $\pi^*(-\frac{1}{2}K_{V_5}) - E$	$\begin{pmatrix} 10 & 5 \\ 5 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\#^2_{15}$	W is the blow-up of $\mathbf{P}^3$ along the intersection of a quadric A and a cubic B such that A is either smooth or reduced but not smooth.	$\pi^*\mathcal{O}_{\mathbf{P}^3}(1)$, $\pi^*\mathcal{O}_{\mathbf{P}^3}(3)-E$	$\begin{pmatrix}4 & 6\\ 6 & 6\end{pmatrix}$	$\begin{pmatrix}1\\ 1\end{pmatrix}$
$\#^2_{16}$	W is the blow-up of V_4 in a conic.	$\pi^*(-\frac{1}{2}K_{V_4})$, $\pi^*(-\frac{1}{2}K_{V_5})-E$	$\begin{pmatrix}8 & 6\\ 6 & 2\end{pmatrix}$	$\begin{pmatrix}1\\ 1\end{pmatrix}$
$\#^2_{17}$	W is the blow-up of a quadric $Q\subset\mathbf{P}^4$ in a an elliptic curve of degree 5.	$\pi^*(-\frac{1}{3}K_Q)$, $\pi^*(-\frac{2}{3}K_Q)-E$	$\begin{pmatrix}6 & 7\\ 7 & 4\end{pmatrix}$	$\begin{pmatrix}1\\ 1\end{pmatrix}$
$\#^2_{18}$	W is a double cover of $\mathbf{P}^1\times\mathbf{P}^2$ branched over a divisor of bidegree $(2,2)$.	$\pi^*\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^2}(1,0)$, $\pi^*\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^2}(0,1)$	$\begin{pmatrix}0 & 4\\ 4 & 2\end{pmatrix}$	$\begin{pmatrix}1\\ 2\end{pmatrix}$
$\#^2_{19}$	W is the blow-up of V_4 along a line.	$\pi^*(-\frac{1}{2}K_{V_4})$, $\pi^*(-\frac{1}{2}K_{V_4})-E$	$\begin{pmatrix}8 & 7\\ 7 & 4\end{pmatrix}$	$\begin{pmatrix}1\\ 1\end{pmatrix}$
$\#^2_{20}$	W is the blow-up of $V_5\subset\mathbf{P}^5$ along a twisted cubic.	$\pi^*(-\frac{1}{2}K_{V_5})$, $\pi^*(-\frac{1}{2}K_{V_5})-E$	$\begin{pmatrix}10 & 7\\ 7 & 2\end{pmatrix}$	$\begin{pmatrix}1\\ 1\end{pmatrix}$
$\#^2_{21}$	W is the blow-up of a quadric $Q\subset\mathbf{P}^4$ along a twisted quartic.	$\pi^*(-\frac{1}{3}K_Q)$, $\pi^*(-\frac{2}{3}K_Q)-E$	$\begin{pmatrix}6 & 8\\ 8 & 6\end{pmatrix}$	$\begin{pmatrix}1\\ 1\end{pmatrix}$
$\#^2_{22}$	W is the blow-up of $V_5\subset\mathbf{P}^6$ along a conic.	$\pi^*(-\frac{1}{2}K_{V_5})$, $\pi^*(-\frac{1}{2}K_{V_5})-E$	$\begin{pmatrix}10 & 8\\ 8 & 4\end{pmatrix}$	$\begin{pmatrix}1\\ 1\end{pmatrix}$
$\#^2_{23}$	W is the blow-up of a quadric $Q\subset\mathbf{P}^4$ along the intersection of two divisors $A\in\lvert i^*\mathcal{O}_{\mathbf{P}^4}(1)\rvert$ and $B\in\lvert i^*\mathcal{O}_{\mathbf{P}^4}(2)\rvert$ with A either smooth or singular.	$\pi^*(-\frac{1}{3}K_Q)$, $\pi^*(-\frac{2}{3}K_Q)-E$	$\begin{pmatrix}6 & 8\\ 8 & 8\end{pmatrix}$	$\begin{pmatrix}1\\ 1\end{pmatrix}$
$\#^2_{24}$	W is a divisor in $\mathbf{P}^2\times\mathbf{P}^2$ of bidegree $(1,2)$.	$\pi^*\mathcal{O}_{\mathbf{P}^2\times\mathbf{P}^2}(1,0)$, $\pi^*\mathcal{O}_{\mathbf{P}^2\times\mathbf{P}^2}(0,1)$	$\begin{pmatrix}2 & 5\\ 5 & 2\end{pmatrix}$	$\begin{pmatrix}2\\ 1\end{pmatrix}$
$\#^2_{25}$	W is the blow-up of $\mathbf{P}^3$ in an elliptic curve which is the intersection of two quadrics.	$\pi^*\mathcal{O}_{\mathbf{P}^3}(1)$, $\pi^*\mathcal{O}_{\mathbf{P}^3}(2)-E$	$\begin{pmatrix}4 & 4\\ 4 & 0\end{pmatrix}$	$\begin{pmatrix}2\\ 1\end{pmatrix}$
$\#^2_{26}$	W is the blow-up of $V_5\subset\mathbf{P}^5$ in a line.	$\pi^*(-\frac{1}{2}K_{V_5})$, $\pi^*(-\frac{1}{2}K_{V_5})-E$	$\begin{pmatrix}10 & 9\\ 9 & 6\end{pmatrix}$	$\begin{pmatrix}1\\ 1\end{pmatrix}$

$\#^2_{27}$	W is the blow-up of $\mathbf{P}^3$ in a twisted cubic.	$\pi^*\mathcal{O}_{\mathbf{P}^3}(1)$, $\pi^*\mathcal{O}_{\mathbf{P}^3}(2) - E$	$\begin{pmatrix} 4 & 5 \\ 5 & 2 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 1 \end{pmatrix}$
$\#^2_{28}$	W is the blow-up of $\mathbf{P}^3$ in a plane cubic.	$\pi^*\mathcal{O}_{\mathbf{P}^3}(1)$, $\pi^*\mathcal{O}_{\mathbf{P}^3}(3) - E$	$\begin{pmatrix} 4 & 9 \\ 9 & 18 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
$\#^2_{29}$	W is the blow-up of a quadric $Q \subset \mathbf{P}^4$ in a conic (the interection of two hyperplanes).	$\pi^*(-\frac{1}{3}K_Q)$, $\pi^*(-\frac{1}{3}K_Q) - E$	$\begin{pmatrix} 6 & 4 \\ 4 & 0 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 1 \end{pmatrix}$
$\#^2_{30}$	W is the blow-up of $\mathbf{P}^3$ in a conic.	$\pi^*\mathcal{O}_{\mathbf{P}^3}(1)$, $\pi^*\mathcal{O}_{\mathbf{P}^3}(2) - E$	$\begin{pmatrix} 4 & 6 \\ 6 & 6 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 1 \end{pmatrix}$
$\#^2_{31}$	W is the blow-up of a quadric $Q \subset \mathbf{P}^4$ in a line.	$\pi^*(-\frac{1}{3}K_Q)$, $\pi^*(-\frac{1}{3}K_Q) - E$	$\begin{pmatrix} 6 & 5 \\ 5 & 2 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 1 \end{pmatrix}$
$\#^2_{32}$	W is a divisor in $\mathbf{P}^2 \times \mathbf{P}^2$ of bidegree $(1,1)$.	$i^*\mathcal{O}_{\mathbf{P}^2\times\mathbf{P}^2}(1,0)$, $i^*\mathcal{O}_{\mathbf{P}^2\times\mathbf{P}^2}(0,1)$	$\begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 2 \end{pmatrix}$
$\#^2_{33}$	W is the blow-up of $\mathbf{P}^3$ along a line.	$\pi^*\mathcal{O}_{\mathbf{P}^3}(1)$, $\pi^*\mathcal{O}_{\mathbf{P}^3}(1) - E$	$\begin{pmatrix} 4 & 3 \\ 3 & 0 \end{pmatrix}$	$\begin{pmatrix} 3 \\ 1 \end{pmatrix}$
$\#^2_{34}$	W is $\mathbf{P}^1 \times \mathbf{P}^2$.	$\pi * \mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^2}(1,0)$, $\pi * \mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^2}(0,1)$	$\begin{pmatrix} 0 & 3 \\ 3 & 2 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 3 \end{pmatrix}$
$\#^2_{35}$	W is the blow-up of $\mathbf{P}^3$ in a point and also denoted by V_7.	$\pi^*\mathcal{O}_{\mathbf{P}^3}(1)$, $\pi^*\mathcal{O}_{\mathbf{P}^3}(1) - E$	$\begin{pmatrix} 4 & 4 \\ 4 & 2 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 2 \end{pmatrix}$
$\#^2_{36}$	W is $\mathbf{P}E$ with $E := \mathcal{O}_{\mathbf{P}_2} \oplus \mathcal{O}_{\mathbf{P}_2}(-2)$.	$\pi^*\mathcal{O}_{\mathbf{P}^2}(1)$, $\mathcal{O}_{\mathbf{P}E}(1)$	$\begin{pmatrix} 2 & 5 \\ 5 & 10 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$
$\#^3_1$	W is a double cover of $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ branched over a divisor of tridegree $(2,2,2)$.	$\pi^*\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1\times\mathbf{P}^1}(1,0,0)$, $\pi^*\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1\times\mathbf{P}^1}(0,1,0)$, $\pi^*\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1\times\mathbf{P}^1}(0,0,1)$	$\begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$
$\#^3_2$	W is a divisor in $\lvert\mathcal{O}_{\mathbf{P}E}(2) \otimes \pi^*\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}(2,3)\rvert$ in $\mathbf{P}E$ with $E := \mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}(1,1)^{\oplus 2}$.	$\pi^*\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}(1,0)$, $\pi^*\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}(0,1)$, $\mathcal{O}_{\mathbf{P}E}(1) \otimes \pi^*\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}(1,1)$	$\begin{pmatrix} 0 & 2 & 3 \\ 2 & 0 & 4 \\ 3 & 4 & 8 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$
$\#^3_3$	W is a divisor in $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^2$ of tridegree $(1,1,2)$.	$i^*\pi^*\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1\times\mathbf{P}^2}(1,0,0)$, $i^*\pi^*\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1\times\mathbf{P}^2}(0,1,0)$, $i^*\pi^*\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1\times\mathbf{P}^2}(0,0,1)$	$\begin{pmatrix} 0 & 2 & 3 \\ 2 & 0 & 3 \\ 3 & 3 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\#^3_4$	W is the blow-up of W^2_{18} (a double cover of $\mathbf{P}^1 \times \mathbf{P}^2$ branched in a divisor of bidegree $(2,2)$) along a smooth fiber of the map $W^2_{18} \longrightarrow \mathbf{P}^2$.	$\pi^*\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^2}(1,0)$, $\pi^*\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^2}(0,1)$, $\pi^*\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^2}(0,1) - E$	$\begin{pmatrix} 0 & 4 & 2 \\ 4 & 2 & 2 \\ 2 & 2 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$
$\#^3_5$	W is the blow-up of W^2_{34} ($\mathbf{P}^1 \times \mathbf{P}^2$) along a curve C of bidegree $(5,2)$ such that the map $C \longrightarrow \mathbf{P}^2$ is an embedding.	$\pi^*\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^2}(1,0)$, $\pi^*\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^2}(0,1)$, $\pi^*\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^2}(1,2) - E$	$\begin{pmatrix} 0 & 3 & 1 \\ 3 & 2 & 5 \\ 1 & 5 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$
$\#^3_6$	W is the blow-up of $\mathbf{P}^3$ along the disjoint union of line and an elliptic curve of degree 4.	$\pi^*\mathcal{O}_{\mathbf{P}^3}(1)$, $\pi^*\mathcal{O}_{\mathbf{P}^3}(1) - E_1$, $\pi^*\mathcal{O}_{\mathbf{P}^3}(2) - E_2$	$\begin{pmatrix} 4 & 3 & 4 \\ 3 & 0 & 2 \\ 4 & 2 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$
$\#^3_7$	W is the blow-up of W^2_{32} (a divisor in $\mathbf{P}^2 \times \mathbf{P}^2$ of bidegree $(1,1)$) along an elliptic curve which the intersection of two divisors in $\lvert i^*\mathcal{O}_{\mathbf{P}^2\times\mathbf{P}^2}(1,1)\rvert$.	$i^*\mathcal{O}_{\mathbf{P}^2\times\mathbf{P}^2}(1,0)$, $i^*\mathcal{O}_{\mathbf{P}^2\times\mathbf{P}^2}(0,1)$, $i^*\mathcal{O}_{\mathbf{P}^2\times\mathbf{P}^2}(1,1) - E$	$\begin{pmatrix} 2 & 4 & 3 \\ 4 & 2 & 3 \\ 3 & 3 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$
$\#^3_8$	W is a divisor in $\lvert\pi_1^*\rho^*\mathcal{O}_{\mathbf{P}^2}(1) \times \pi_2^*\mathcal{O}_{\mathbf{P}^2}(1)\rvert$ in $\mathbf{F}_1 \times \mathbf{P}^2$.	$\pi_1^*\rho^*\mathcal{O}_{\mathbf{P}^2}(1)$, $\pi_1^*(\rho^*\mathcal{O}_{\mathbf{P}^2}(1) - E)$, $\pi_2^*\mathcal{O}_{\mathbf{P}^2}(1)$	$\begin{pmatrix} 2 & 2 & 5 \\ 2 & 0 & 3 \\ 5 & 3 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$
$\#^3_9$	W is the blow-up of a cone $W_4 \subset \mathbf{P}^6$ over the Veronese surface $R_4 \subset \mathbf{P}^5$ with center in a disjoint union of the vertex and a quartic C in $R_4 \cong \mathbf{P}^2$. This is blow-up agrees with $\mathbf{P}E$ with $E := \mathcal{O}_{\mathbf{P}^2} \oplus \mathcal{O}_{\mathbf{P}^2}(2)$) and R_4 corresponds to the zero section.	$\pi^*\rho^*\mathcal{O}_{\mathbf{P}^2}(1)$, $\pi^*\rho^*\mathcal{O}_{\mathbf{P}^2}(2) \otimes \pi^*\mathcal{O}_{\mathbf{P}E}$, $\pi^*\rho^*\mathcal{O}_{\mathbf{P}^2}(4) \otimes \pi^*\mathcal{O}_{\mathbf{P}E}(1) - E$	$\begin{pmatrix} 2 & 5 & 5 \\ 5 & 10 & 12 \\ 5 & 12 & 10 \end{pmatrix}$	$\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

$\#^3_{10}$	W is the blowup of a quadric $Q \subset \mathbf{P}^4$ in two disjoint conics.	$\pi^* i^* \mathcal{O}_{\mathbf{P}^4}(1)$, $\pi^* i^* \mathcal{O}_{\mathbf{P}^4}(1) - E_1$, $\pi^* i^* \mathcal{O}_{\mathbf{P}^4}(1) - E_2$	$\begin{pmatrix} 6 & 4 & 4 \\ 4 & 0 & 2 \\ 4 & 2 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$
$\#^3_{11}$	W is the blowup of W^2_{35} (V_7) in an elliptic curve which is the intersection of two divisors in $\lvert -\frac{1}{2}K_{V_7} \rvert$.	$\pi^* \rho^* \mathcal{O}_{\mathbf{P}^3}(1)$, $\pi^*(\rho^* \mathcal{O}_{\mathbf{P}^3}(1) - E_1)$, $\pi^*(\rho^* \mathcal{O}_{\mathbf{P}^3}(2) - E_1) - E_2$	$\begin{pmatrix} 4 & 4 & 4 \\ 4 & 2 & 3 \\ 4 & 3 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$
$\#^3_{12}$	W is the blow-up of $\mathbf{P}^3$ along a disjoint union of a line and a twisted cubic.	$\pi^* \mathcal{O}_{\mathbf{P}^3}(1)$, $\pi^* \mathcal{O}_{\mathbf{P}^3}(1) - E_1$, $\pi^* \mathcal{O}_{\mathbf{P}^3}(2) - E_2$	$\begin{pmatrix} 4 & 3 & 5 \\ 3 & 0 & 3 \\ 5 & 3 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$
$\#^3_{13}$	W is the blow-up of W^2_{32} (a divisor in $\mathbf{P}^2 \times \mathbf{P}^2$ of bidegree $(1,1)$) along a curve C of bidegree $(2,2)$ such that both maps $C \to \mathbf{P}^2$ are embeddings.	$\pi^* i^* \mathcal{O}_{\mathbf{P}^2 \times \mathbf{P}^2}(1,0)$, $\pi^* i^* \mathcal{O}_{\mathbf{P}^2 \times \mathbf{P}^2}(1,0)$, $\pi^* i^* \mathcal{O}_{\mathbf{P}^2 \times \mathbf{P}^2}(2,2) - E$	$\begin{pmatrix} 2 & 4 & 10 \\ 4 & 2 & 10 \\ 10 & 10 & 30 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$
$\#^3_{14}$	W is the blowup of $\mathbf{P}^3$ along a cubic lying in a plane and a point not contained in this plane.	$\pi^* \mathcal{O}_{\mathbf{P}^3}(1)$, $\pi^* \mathcal{O}_{\mathbf{P}^3}(3) - E_1$, $\pi^* \mathcal{O}_{\mathbf{P}^3}(1) - E_2$,	$\begin{pmatrix} 4 & 9 & 4 \\ 9 & 18 & 9 \\ 4 & 9 & 2 \end{pmatrix}$	$\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$
$\#^3_{15}$	W is the blow-up of a quadric $Q \subset \mathbf{P}^4$ along a disjoint union of a line and a conic.	$\pi^* i^* \mathcal{O}_{\mathbf{P}^4}(1)$, $\pi^* i^* \mathcal{O}_{\mathbf{P}^4}(1) - E_1$, $\pi^* i^* \mathcal{O}_{\mathbf{P}^4}(1) - E_2$	$\begin{pmatrix} 6 & 5 & 4 \\ 5 & 2 & 3 \\ 4 & 3 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$
$\#^3_{16}$	W is the blow-up of W^2_{32} (V_7, the blow-up of $\mathbf{P}^3$ in a point x) along the proper transform of a twisted cubic through x.	$\pi^* \rho^* \mathcal{O}_{\mathbf{P}^3}(1)$, $\pi^*(\rho^* \mathcal{O}_{\mathbf{P}^3}(1) - E_1)$, $\pi^* \rho^* \mathcal{O}_{\mathbf{P}^3}(1) - E_2$	$\begin{pmatrix} 4 & 4 & 5 \\ 4 & 2 & 6 \\ 5 & 6 & 2 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$
$\#^3_{17}$	W is a divisor of tri-degree $(1,1,1)$ in $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^2$.	$i^* \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1}(1,0,0)$, $i^* \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1}(0,1,0)$, $i^* \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1}(0,0,1)$	$\begin{pmatrix} 0 & 2 & 3 \\ 2 & 0 & 3 \\ 3 & 3 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

$\#^3_{18}$	W is the blow-up of $\mathbf{P}^3$ in a line and a conic.	$\pi^*\mathcal{O}_{\mathbf{P}^3}(1)$, $\pi^*\mathcal{O}_{\mathbf{P}^3}(1)-E_1$, $\pi^*\mathcal{O}_{\mathbf{P}^3}(2)-E_2$	$\begin{pmatrix}4&3&6\\3&0&4\\6&4&6\end{pmatrix}$	$\begin{pmatrix}1\\1\\1\end{pmatrix}$
$\#^3_{19}$	W is the blow-up of a quadric $Q\subset\mathbf{P}^4$ in two non-colinear points.	$\pi^*i^*\mathcal{O}_{\mathbf{P}^4}(1)$, $\pi^*i^*\mathcal{O}_{\mathbf{P}^4}(1)-E_1$, $\pi^*i^*\mathcal{O}_{\mathbf{P}^4}(1)-E_2$	$\begin{pmatrix}6&6&6\\6&4&6\\6&6&4\end{pmatrix}$	$\begin{pmatrix}-1\\2\\2\end{pmatrix}$
$\#^3_{20}$	W is the blow-up of a quadric $Q\subset\mathbf{P}^4$ along the disjoint union of two lines.	$\pi^*i^*\mathcal{O}_{\mathbf{P}^4}(1)$, $\pi^*i^*\mathcal{O}_{\mathbf{P}^4}(1)-E_1$, $\pi^*i^*\mathcal{O}_{\mathbf{P}^4}(1)-E_2$	$\begin{pmatrix}6&5&5\\5&2&4\\5&4&2\end{pmatrix}$	$\begin{pmatrix}1\\1\\1\end{pmatrix}$
$\#^3_{21}$	W is the blow-up of W^2_{34} ($\mathbf{P}^1\times\mathbf{P}^2$) in a curve of bidegree $(2,1)$.	$\pi^*\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^2}(1,0)$, $\pi^*\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^2}(0,1)$, $\pi^*\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^2}(2,1)$	$\begin{pmatrix}0&3&1\\3&2&7\\1&7&2\end{pmatrix}$	$\begin{pmatrix}0\\2\\1\end{pmatrix}$
$\#^3_{22}$	W is the blow-up of W^2_{34} ($\mathbf{P}^1\times\mathbf{P}^2$) in a conic in $\{x\}\times\mathbf{P}^2$.	$\pi^*\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^2}(1,0)$, $\pi^*\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^2}(0,1)$, $\pi^*\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^2}(1,2)-E$	$\begin{pmatrix}0&3&6\\3&2&5\\6&5&10\end{pmatrix}$	$\begin{pmatrix}1\\1\\1\end{pmatrix}$
$\#^3_{23}$	W is the blow-up of W^2_{32} (the blow-up of $\mathbf{P}^3$ in a point x) along the proper transform of a conic through x.	$\pi^*\rho^*\mathcal{O}_{\mathbf{P}^3}(1)$, $\pi^*(\rho^*\mathcal{O}_{\mathbf{P}^3}(1)-E_1)$, $\pi^*(\rho^*\mathcal{O}_{\mathbf{P}^3}(1)-E_1)-E_2$	$\begin{pmatrix}4&4&6\\4&2&5\\6&5&6\end{pmatrix}$	$\begin{pmatrix}1\\1\\1\end{pmatrix}$
$\#^3_{24}$	W is the blow-up of W^2_{32} (a divisor in $\mathbf{P}^2\times\mathbf{P}^2$ of bidegree $(1,1)$) in a fiber the projection $W^2_{32}\longrightarrow\mathbf{P}^2$ onto the second factor.	$\pi^*i^*\mathcal{O}_{\mathbf{P}^2\times\mathbf{P}^2}(1,0)$, $\pi^*i^*\mathcal{O}_{\mathbf{P}^2\times\mathbf{P}^2}(0,1)$, $\pi^*i^*\mathcal{O}_{\mathbf{P}^2\times\mathbf{P}^2}(0,1)-E$	$\begin{pmatrix}2&4&3\\4&2&2\\3&2&0\end{pmatrix}$	$\begin{pmatrix}2\\1\\1\end{pmatrix}$
$\#^3_{25}$	W is the blow-up of $\mathbf{P}^3$ along the disjoint union of two lines or, equivalently, $\mathbf{P}E$ with $E:=\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}(1,0)\oplus\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}(0,1)$.	$\pi^*\mathcal{O}_{\mathbf{P}^3}(1)$, $\pi^*\mathcal{O}_{\mathbf{P}^3}(1)-E_1$, $\pi^*\mathcal{O}_{\mathbf{P}^3}(1)-E_2$	$\begin{pmatrix}4&3&3\\3&0&2\\3&2&0\end{pmatrix}$	$\begin{pmatrix}2\\1\\1\end{pmatrix}$
$\#^3_{26}$	W is the blow-up of $\mathbf{P}^3$ in the disjoint union of a point and line.	$\pi^*\mathcal{O}_{\mathbf{P}^3}(1)$, $\pi^*\mathcal{O}_{\mathbf{P}^3}(1)-E_1$, $\pi^*\mathcal{O}_{\mathbf{P}^3}(1)-E_2$	$\begin{pmatrix}4&4&3\\4&2&3\\3&3&0\end{pmatrix}$	$\begin{pmatrix}1\\2\\1\end{pmatrix}$

$\#^3_{27}$	W is $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$.	$\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1\times\mathbf{P}^1}(1,0,0)$, $\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1\times\mathbf{P}^1}(0,1,0)$, $\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1\times\mathbf{P}^1}(0,0,1)$	$\begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$
$\#^3_{28}$	W is $\mathbf{P}^1 \times \mathbf{F}_1$ or equivalently the blow-up of $\mathbf{P}^1 \times \mathbf{P}^2$ in $\mathbf{P}^1 \times \{x\}$.	$\pi^*\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^2}(1,0)$, $\pi^*\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^2}(0,1)$, $\pi^*\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^2}(0,1) - E$	$\begin{pmatrix} 0 & 3 & 2 \\ 3 & 2 & 2 \\ 2 & 2 & 0 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$
$\#^3_{29}$	W is the blowup of W^2_{35} (the blow-up of $\mathbf{P}^3$ in a point) in a line in the exceptional divisor.	$\pi^*\rho^*\mathcal{O}_{\mathbf{P}^3}(1)$, $\pi^*(\rho^*\mathcal{O}_{\mathbf{P}^3}(1) - E_1)$, $\pi^*\rho^*\mathcal{O}_{\mathbf{P}^3}(1) - E_2$	$\begin{pmatrix} 4 & 4 & 4 \\ 4 & 2 & 3 \\ 4 & 3 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$
$\#^3_{30}$	W is the blow-up of W^2_{35} (the blow-up of $\mathbf{P}^3$ in a point x) along a the proper transform of a line through x.	$\pi^*\rho^*\mathcal{O}_{\mathbf{P}^3}(1)$, $\pi^*(\rho^*\mathcal{O}_{\mathbf{P}^3}(1) - E_1)$, $\pi^*(\rho^*\mathcal{O}_{\mathbf{P}^3}(1) - E_1) - E_2$	$\begin{pmatrix} 4 & 4 & 3 \\ 4 & 2 & 2 \\ 3 & 2 & 0 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$
$\#^3_{31}$	W is $\mathbf{P}E$ with $E := \mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}(-1,-1)$.	$\pi^*\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}(1,0))$, $\pi^*\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}(0,1)$, $\mathcal{O}_{\mathbf{P}E}(1)$	$\begin{pmatrix} 0 & 2 & 3 \\ 2 & 0 & 3 \\ 3 & 3 & 6 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

C Fano 3–folds whose ample anticanonical bundle is not very ample

According to [IP99, Theorem 2.4.5, Theorem 2.1.16, and the Remarks preceeding Section 12.3] if W is a Fano 3 with $-K_W$ is not very ample, then W is one of the following:

1. a double cover of $\mathbf{P}^3$ branched along a divisor of degree 6,
2. a double cover of a quadric branched along a divisor of degree 8,
3. V_1, a double cover of $C \subset \mathbf{P}^6$, a cone over the Veronese surface in $\mathbf{P}^5$, branched along a cubic hypersurface in C not passing through the vertex, or a hypersurface of degree 6 in the weighted projective space $\mathbf{P}(1,1,1,2,3)$,
4. the blow-up of V_1 along an elliptic curve which is an intersection of two divisors in $|-\frac{1}{2}K_{V_1}|$,
5. a double cover of $\mathbf{P}^1 \times \mathbf{P}^2$ branched along a divisor of bidegree $(2,4)$,
6. the blow-up of V_2 along an elliptic curve which is an intersection of two divisors in $|-\frac{1}{2}K_{V_2}|$ (V_2 is a double cover of $\mathbf{P}^3$ branched along divisor of degree 4),
7. $\mathbf{P}^1 \times S_2$, or

8. $\mathbf{P}^1 \times S_1$.

Here S_ℓ is a del Pezzo surface of degree ℓ. The double cover of a quadric branched along a divisor of degree 8 can be deformed to a quartic in $\mathbf{P}^3$, for which $-K_W$ is, of course, very ample.

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