Hermitian Yang–Mills metrics on reflexive sheaves over asymptotically cylindrical Kähler manifolds

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Abstract

We prove an analogue of the Donaldson–Uhlenbeck–Yau theorem for asymptotically cylindrical (ACyl) Kähler manifolds: If $\mathscr{E}$ is a reflexive sheaf over an ACyl Kähler manifold, which is asymptotic to a μ–stable holomorphic vector bundle, then it admits an asymptotically translation-invariant projectively Hermitian Yang–Mills metric (with curvature in L^2_{loc} across the singular set). Our proof combines the analytic continuity method of Uhlenbeck and Yau [UY86] with the geometric regularization scheme introduced by Bando and Siu [BS94].

1 Introduction

In this paper we construct (singular) projectively Hermitian Yang–Mills (**P**HYM) metrics over a certain class of complete non-compact Kähler manifolds.

In the compact case this problem has been extensively studied. Its solution provides a particularly beautiful example of the relation between canonical metrics and algebro-geometric notions of stability: a holomorphic vector bundle over a compact Kähler admits a **P**HYM metric if and only if it is μ–polystable. This was first proved for curves by Narasimhan and Seshadri [NS65], for algebraic surfaces by Donaldson [Don85], and for arbitrary compact Kähler manifolds by Uhlenbeck and Yau [UY86].

It is an interesting and important question to ask: under which hypothesis does a holomorphic vector bundle over a complete non-compact Kähler manifolds admit a **P**HYM metric?[1] The answer to this question is not completely understood, but a number of partial results have been obtained. For asymptotically conical Kähler manifolds, Bando proved the existence of **P**HYM metrics on holomorphic vector bundles which are flat at infinity [Ban93]. Ni and Ren [NR01] proved that a holomorphic vector bundle over a complete non-compact Kähler manifold with a spectral gap admits a **P**HYM metric if and only if it admits a metric whose failure to be **P**HYM is in L^p for $p > 1$ (using an argument similar to Donaldson's solution of the Dirichlet problem for the **P**HYM equation [Don92]). Ni [Ni02] showed that the same conclusion holds, for example, if the Kähler

[1] This question was also raised in Yau's 2015 Shanks Lecture [Yau15, p. 66].

manifold satisfies a L^2 Sobolev inequality and $p \in [1, n/2)$, or if it is non-parabolic (i.e., admits a positive Green's function) and $p = 1$.

Main result In this article we concentrate on the asymptotically cylindrical case, and in view of the applications we have in mind we work with reflexive sheaves (not just holomorphic vector bundles).

Theorem 1.1. *Let V be an asymptotically cylindrical (ACyl) Kähler manifold with asymptotic cross-section D. Let $\mathscr{E}_D$ be a μ–stable vector bundle over D, and $\mathscr{E}$ a reflexive sheaf asymptotic to $\mathscr{E}_D$.*

*Then there exists an asymptotically translation-invariant Hermitian metric H on $\mathscr{E}$ which satisfies the projective Hermitian Yang–Mills (**P**HYM) equation*

$$K_H \coloneqq i\Lambda F_H - \frac{\operatorname{tr}(i\Lambda F_H)}{\operatorname{rk}\mathscr{E}} \cdot \mathrm{id}_{\mathscr{E}} = 0, \tag{1.2}$$

and $|F_H| \in L^2_{\mathrm{loc}}(V)$.

Remark 1.3. A **P**HYM metric H on $\mathscr{E}$ is Hermitian Yang–Mills (HYM) if and only if the induced metric h on $\det \mathscr{E}$ is HYM, that is, $i\Lambda F_h = \frac{\operatorname{tr}(i\Lambda F_H)}{\operatorname{rk}\mathscr{E}}$ is constant. Every asymptotically translation-invariant line bundle over an ACyl Kähler manifold has a HYM metric; however, this metric will typically not be asymptotically translation invariant. See Section 2.3 for a detailed discussion.

Remark 1.4. The definition of asymptotically cylindrical Kähler manifolds we work with is given in Definition 2.1; it includes being asymptotically fibred.

Remark 1.5. The question of the existence of HYM metrics on holomorphic bundles (with trivial determinant) over ACyl Calabi–Yau 3–folds was studied earlier by Sá Earp [Sá 15] (using the Yang–Mills heat flow). Our result improves on his in that we consider general ACyl Kähler manifolds and handle reflexive sheaves; moreover, we give a complete proof of the exponential decay to a **P**HYM metric over D (which is crucial for applications).

Remark 1.6. In dimension four, there is prior work on the relation between ASD instantons and holomorphic vector bundles over cylindrical manifolds by Guo [Guo96] and Owens [Owe01].

Remark 1.7. Theorem 1.1 does not make any statement about the behavior of H near singularities. Jacob, Sá Earp, and Walpuski [JSW18] and Chen and Sun [CS17] have studied this behavior in the case of isolated singularities.

Examples and applications There are plenty of examples of ACyl Kähler manifolds and reflexive sheaves on them. Given any smooth projective variety Z containing a smooth divisor D and fibred by $|D|$, $V \coloneqq Z \backslash D$ can be given the structure of an ACyl Kähler manifold [HHN15, Section 4.2, Part 1]. Theorem 1.1 can be applied to any holomorphic vector bundle $\mathscr{E}$ on Z such that $\mathscr{E}|_D$ is μ–stable. One often wants to construct $\mathscr{E}$ by extending a holomorphic vector bundle $\mathscr{E}_D$ on D to all of Z. This can always be achieved with $\mathscr{E}$ being a reflexive sheaf—by first extending $\mathscr{E}_D$ as a torsion-free sheaf and then taking the reflexive hull. Whether or not this extension can be arranged to be a holomorphic vector bundle is a subtle question. This is one of the reasons why it is desirable to allow reflexive sheaves.

ACyl Calabi–Yau 3–folds are an important ingredient in the construction of twisted connected sum G_2–manifolds [Kov03; KL11; CHNP15]. Building on [Sá 15], Sá Earp and the second named author gave a construction of a class of Yang–Mills connections, called G_2–instantons, over such twisted connected sums [SW15]; see [Wal15] for a concrete example. We hope that the current work will be a first step towards the construction of singular G_2–instantons on twisted connected sums. G_2–instantons play a central role in Donaldson and Thomas' vision of gauge theory in higher dimensions [DT98], and understanding singularities and their formation is an important part of making their ideas rigorous; see, e.g., [Wal13; Wal17; HW15].

Proof idea We first prove Theorem 1.1 for holomorphic vector bundles. After a suitiable choice of an initial Hermitian metric H_0 on $\mathscr{E}$, we construct a **P**HYM metric using the Uhlenbeck–Yau continuity method. The difficult part is the a priori C^0 estimate on the endomorphism s relating H_0 and the Hermitian metric $H_t = H_0 e^s$ along the continuity path. Unlike in [Ban93; Ni02], a solution to the Poisson equation $\Delta f = |K_{H_0}|$ can not act as a barrier, since on V such a solution does not have exponential decay—in fact, it decreases linearly along the cylindrical end. Instead, we use an adaptation to our setup of Sá Earp's argument in [Sá 15]: his proof first exploits the barrier to obtain a bound of the form $\|s\|_{L^\infty}^3 \lesssim \|s\|_{L^2}^2$, and then uses the Donaldson functional on transverse slices along the cylindrical end to show that $\|s\|_{L^2} \lesssim \|s\|_{L^\infty}$. Besides the construction of the initial Hermitian metric H_0, this is the point at which μ–stability enters into the proof. To prove a priori exponential decay bounds we use ideas of Haskins, Hein and Nordström [HHN15].

Once Theorem 1.1 is established for holomorphic vector bundles, we prove the general case for a reflexive sheaf $\mathscr{E}$ following a geometric regularization scheme, introduced by Bando and Siu [BS94], based on approximating $\mathscr{E}$ and V by a holomorphic vector bundle and a family of ACyl Kähler metrics on a blow-up of V. The main difficulty is controlling the barrier f as the metrics degenerate. Once f is controlled, the C^0 bound on compact subsets away from the singular set of $\mathscr{E}$ follows, and the arguments from the holomorphic vector bundle case can be applied directly.

Conventions We denote by $c > 0$ a generic constant, which depends only on V, $\mathscr{E}$, and the reference metric H_0 constructed in Section 3. Its value might change from one occurrence to the next. Should c depend on further data we indicate this by a subscript. We write $x \lesssim y$ for $x \leqslant cy$ and $x \asymp y$ for $c^{-1}y \leqslant x \leqslant cy$. $O(x)$ denotes a quantity y with $|y| \lesssim x$.

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2 ACyl Kähler manifolds

In this section we briefly introduce some notation, recall the necessary linear analysis, and provide the details promised in Remark 1.3.

Definition 2.1. Let (D, g_D, I_D) be a compact Kähler manifold. A Kähler manifold (V, g, I) is called **asymptotically cylindrical (ACyl)** with asymptotic cross-section (D, g_D, I_D) if there exists a constant $\delta_V > 0$, a compact subset $K \subset V$ and a diffeomorphism $\pi\colon V \backslash K \to (1, \infty) \times S^1 \times D$ such that

$$|\nabla^k(\pi_* g - g_\infty)| + |\nabla^k(\pi_* I - I_\infty)| = O(e^{-\delta_V \ell}),$$

for all $k \in \mathbf{N}_0$, with

$$g_\infty := \mathrm{d}\ell^2 \oplus \mathrm{d}\theta^2 \oplus g_D \quad \text{and} \quad I_\infty = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \oplus I_D.$$

Here (ℓ, θ) are the canonical coordinates on $(0, \infty) \times S^1$. The connection ∇ and norms $|\cdot|$ are both taken to be the ones induced by g_∞. Moreover, we assume that the map $V \backslash K \to (1, \infty) \times S^1$ is holomorphic.

In what follows, we suppose that an ACyl Kähler manifold V with asymptotic cross-section D has been fixed. By slight abuse of notation we denote by $\ell\colon V \to [0, \infty)$ a smooth extension of $\ell \circ \pi\colon V \backslash K \to (1, \infty)$ such that $\ell \leqslant 1$ on K. Given $L > 1$, we define the truncated manifold

$$V_L := \ell^{-1}([0, L]).$$

Given $z = (L, \theta) \in (1, \infty) \times S^1$, we set

$$D_z := \pi^{-1}(\{(L, \theta)\} \times D). \tag{2.2}$$

2.1 Reflexive sheaves and Hermitian metrics

Definition 2.3. Let $\mathscr{E}_D = (E_D, \bar\partial_D)$ be a holomorphic vector bundle over D. Let $\mathscr{E}$ be a reflexive sheaf over V with singular set $S := \operatorname{sing}(\mathscr{E})$ and underlying smooth vector bundle $E \to V \backslash S$. We say that $\mathscr{E}$ is **asymptotic** to $\mathscr{E}_D$ if the following hold:

- There exists a constant $L_0 \geqslant 2$ such that $S \subset V_{L_0 - 1}$. In particular, $E|_{V \backslash V_{L_0}}$ has a $\bar\partial$–operator.
- Denote by $\mathscr{E}_\infty = (E_\infty, \bar\partial_\infty)$ the pullback of $\mathscr{E}_D = (E_D, \bar\partial_D)$ to $(L_0, \infty) \times S^1 \times D$. Choose an auxiliary Hermitian metric on E_D and pull it back to E_∞.[2] There exists a bundle isomorphism $\bar\pi\colon E|_{V \backslash V_{L_0}} \to E_\infty$ covering π and a constant $\delta_{\mathscr{E}} > 0$ such that

  $$|\nabla^k(\bar\pi_* \bar\partial - \bar\partial_\infty)| = O(e^{-\delta_{\mathscr{E}} \ell}),$$

  for all $k \in \mathbf{N}_0$ and $\ell \geqslant L_0$.

[2] The definition is insensitive to the precise choice, since D is compact.

We say that $(\mathscr{E},\bar\partial)$ is **asymptotically translation-invariant** if it is asymptotic to some holomorphic vector bundle over D.

Definition 2.4. Let $\mathscr{E}$ be a reflexive sheaf over V asymptotic to $\mathscr{E}_D$. Let H_D be a Hermitian metric on E_D. Denote by H_∞ the pullback of H_D to $\mathscr{E}_\infty$. A **Hermitian metric** on $\mathscr{E}$ is a Hermitian metric H on $\mathscr{E}|_{V\backslash S}$. We say that it is **asymptotic** to H_D if there exist a constant $\delta_H > 0$ such that

$$|\nabla^k(\bar\pi_*H - H_\infty)| = O(e^{-\delta_H \ell})$$

for all $k \in \mathbf{N}_0$ and $\ell \geqslant L_0$. (We take the background metric, used in the comparison, to be H_∞.) We say that H is **asymptotically translation-invariant** if it is asymptotic to some Hermitian metric H_D.

Given a Hermitian metric H on a holomorphic vector bundle $(\mathscr{E},\bar\partial)$, there exists a unique connection A_H, called the **Chern connection**, which preserves the Hermitian metric and satisfies $\nabla^{0,1}_{A_H} = \bar\partial$; see, e.g., [Bal06, Theorem 3.18]. We denote the curvature of this connection by F_H.

Definition 2.5. A Hermitian metric H on a reflexive sheaf $\mathscr{E}$ is called **projectively Hermitian Yang–Mills (PHYM)** if $K_H \in C^\infty(V\backslash S, i\mathfrak{su}(E,H))$ defined by

$$K_H \coloneqq i\Lambda F_H - \frac{\operatorname{tr}(i\Lambda F_H)}{\operatorname{rk}\mathscr{E}} \cdot \mathrm{id}_{\mathscr{E}}$$

vanishes.

2.2 Linear analysis

In the subsequent sections we need a few results about linear analysis on ACyl Kähler manifolds. We will simply state the required results and sketch their proofs. For a nice review of linear analysis on ACyl manifolds we refer the reader to [HHN15, Section 2.1]; see also Maz'ya and Plamenevskiĭ [MP78] and Lockhart and McOwen [LM85].

Fix a holomorphic vector bundle $\mathscr{E}$ asymptotic to $\mathscr{E}_D$ and a Hermitian metric H asymptotic to H_D.

Definition 2.6. For $k \in \mathbf{N}$, $\alpha \in (0,1)$ and $\delta \in \mathbf{R}$, define

$$C^{k,\alpha}_\delta(V) \coloneqq \left\{ f \in C^{k,\alpha}(V) : \|f\|_{C^{k,\alpha}_\delta} < \infty \right\},$$

with

$$\|\cdot\|_{C^{k,\alpha}_\delta} \coloneqq \|e^{\delta\ell}\cdot\|_{C^{k,\alpha}},$$

and set

$$C^\infty_\delta(V) \coloneqq \bigcap_{k\in\mathbf{N}} C^{k,\alpha}_\delta(V).$$

Similarly, we define $C^{k,\alpha}_\delta(V, i\mathfrak{su}(E,H))$ and $C^\infty_\delta(V, i\mathfrak{su}(E,H))$.

Proposition 2.7. *For $0 < \delta \ll_D 1$, the linear map $C^{k+2,\alpha}_\delta(V) \oplus \mathbf{R} \to C^{k,\alpha}_\delta(V)$ defined by*

$$(f, A) \mapsto \Delta f - A\Delta\ell$$

is an isomorphism.

Proof. This is [HHN15, Proposition 2.7] together with the observation that

$$\int_V \Delta\ell = -\mathrm{vol}(S^1 \times D). \qquad \square$$

Definition 2.8. A holomorphic vector bundle $(\mathscr{E}_D, \bar\partial)$ over D is**simple** if every holomorphic endomorphisms of $\mathscr{E}_D$ is a homothety, that is: $H^0(\mathscr{E}\mathrm{nd}(\mathscr{E}_D)) = \mathbf{C} \cdot \mathrm{id}_{\mathscr{E}_D}$.

Proposition 2.9. *If H_D is HYM, $\mathscr{E}_D$ is simple and $|\delta| \ll_{H_D} 1$, then the linear operator*

$$\nabla^*_{H_0}\nabla_{H_0} : C^{k+2,\alpha}_\delta(V, i\mathfrak{su}(E,H)) \to C^{k,\alpha}_\delta(V, i\mathfrak{su}(E,H))$$

is Fredholm of index zero.

Proof. We use the theory explained in [HHN15, Section 2.1]. The linear operator $\nabla^*_{H_0}\nabla_{H_0}$ is asymptotic to the translation-invariant linear operator

$$-\partial_\ell^2 - \partial_\theta^2 + \nabla^*_{H_D}\nabla_{H_D}$$

acting on sections of $i\mathfrak{su}(E_\infty, H_\infty)$. Since H_D is **P**HYM,

$$\frac{1}{2}\nabla^*_{H_D}\nabla_{H_D} = \partial^*_{H_D}\partial_{H_D} = \bar\partial^*_{\mathscr{E}_D}\bar\partial_{\mathscr{E}_D}.$$

The latter is invertible because $\mathscr{E}_D$ is simple. Consequently, the spectrum of $-\partial_\theta^2 + \nabla^*_{H_D}\nabla_{H_D}$ is contained in $[\lambda_D, \infty)$, for some $\lambda_D > 0$. This implies the Fredholm property for $|\delta| < \sqrt{\lambda_D}$ by [HHN15, Proposition 2.4]. Since $\nabla^*_{H_D}\nabla_{H_D}$ is formally self-adjoint and 0 is not a critical weight, the index is zero; cf. [LM85, Theorem 7.4]. $\square$

2.3 Hermitian Yang–Mills metrics on line bundles

Proposition 2.10. *Let $\mathscr{L}$ be a line bundle asymptotic to $\mathscr{L}_D$ and denote by h_D a Hermitian metric on $\mathscr{L}_D$ with*

$$i\Lambda F_{h_D} = \lambda := \frac{2\pi \cdot \deg(\mathscr{L}_D)}{(n-2)! \cdot \mathrm{vol}(D)}.$$ [3]

Then there exists a unique Hermitian metric h_0 asymptotic to h_D and $A \in \mathbf{R}$ such that $h := h_0 e^{-A\ell}$ satisfies

$$i\Lambda F_h = \lambda.$$

[3] Such a Hermitian metric exists and is unique up to multiplication by a positive constant; see, e.g., [LT95, Corollary 2.1.6].

Proof. Let h_{-1} be any Hermitian metric asymptotic to h_D. We have

$$\lambda - i\Lambda F_{h_{-1}} \in C^\infty_\delta(V).$$

By Proposition 2.7 there is a unique pair $f \in C^\infty_\delta(V)$ and $A \in \mathbf{R}$ such that

$$\Delta(f - A\ell) = \lambda - i\Lambda F_{h_{-1}}.$$

The proposition follows with $h_0 := h_{-1}e^f$. $\square$

The number $A(\mathscr{L})$ defined by Proposition 2.10 is an invariant of the asymptotically translation-invariant line bundle $\mathscr{L}$. It can be computed as

$$A(\mathscr{L}) := \frac{1}{\mathrm{vol}(S^1 \times D)} \int_V \lambda - i\Lambda F_h$$

with h denoting *any* Hermitian metric asymptotic to some h_D as in Proposition 2.10. It is closely related to the first Chern class: if $\mathscr{L}_1$ and $\mathscr{L}_2$ are both asymptotic to $\mathscr{L}_D$, then $c_1(\mathscr{L}_1) - c_1(\mathscr{L}_2) \in H^2_c(V)$ and

$$A(\mathscr{L}_1) - A(\mathscr{L}_2) = \frac{2\pi \cdot \langle (c_1(\mathscr{L}_1) - c_1(\mathscr{L}_2)) \cup [\omega]^{n-1}, [V]\rangle}{(n-1)! \cdot \mathrm{vol}(S^1 \times D)}.$$

It follows from the above that $\mathscr{E}$ as in Theorem 1.1 admits an asymptotically translation invariant HYM metric if and only if $A(\det \mathscr{E}) = 0$.

3 The Uhlenbeck–Yau continuity method

In this section we begin the proof of Theorem 1.1 in the case when $\mathscr{E}$ is a holomorphic vector bundle. We use the continuity method introduced by Uhlenbeck and Yau [UY86]; see also Lübke and Teleman's beautiful books [LT95; LT06].

We fix some

$$0 < \delta < \min\left\{\delta_V, \delta_{\mathscr{E}}, \sqrt{\lambda_D}\right\}$$

and will shortly construct a background Hermitian metric H_0 on $\mathscr{E}$ which is asymptotically translation-invariant and satisfies

$$K_{H_0} \in C^\infty_\delta(V, i\mathfrak{su}(E, H_0)). \tag{3.1}$$

Given such an H_0, we define a map

$$\mathfrak{L}\colon\ C^\infty_\delta(V, i\mathfrak{su}(E, H_0)) \times [0,1] \to C^\infty_\delta(V, i\mathfrak{su}(E, H_0))$$

by

$$\mathfrak{L}(s,t) := \mathrm{Ad}(e^{s/2})K_{H_0e^s} + t \cdot s.\text{[4]}$$

[4]The prefactor $\mathrm{Ad}(e^{s/2})$ is needed because $K_{H_0e^s}$ need not be H_0–self-adjoint.

Set

$$I := \left\{t \in [0,1] : \mathfrak{L}(s,t) = 0 \text{ for some } s \in C^\infty_\delta(V, i\mathfrak{su}(E,H_0))\right\}.$$

We will show that $1 \in I$, $I \cap (0,1]$ is open and I is closed; hence, $I = [0,1]$. Since $\mathfrak{L}(s,0) = 0$ precisely means that $H = H_0 e^s$ satisfies (1.2), this will prove Theorem 1.1 when $\mathscr{E}$ is a holomorphic vector bundle.

Proposition 3.2. *There exists an asymptotically translation-invariant Hermitian metric H_0 on $\mathscr{E}$ satisfying (3.1) and an $s \in C^\infty_\delta(V, i\mathfrak{su}(E,H_0))$ such that $\mathfrak{L}(s,1) = 0$.*

Proof. We use a trick discovered by Lübke and Teleman [LT95, Lemma 3.2.1]. By the Donaldson–Uhlenbeck–Yau theorem [Don85; UY86; Don87] there exists a PHYM metric H_D on $\mathscr{E}_D$. One can easily construct a Hermitian metric H_{-1} asymptotic to H_D (at rate $\delta_{H_{-1}} = \delta$) which satisfies

$$\kappa := K_{H_{-1}} \in C^\infty_\delta(V, i\mathfrak{su}(E,H_{-1})).$$

The Hermitian metric

$$H_0 := H_{-1}e^\kappa$$

is asymptotic to H_D (at rate $\delta_{H_0} = \delta$). We have (3.1), and $\kappa \in C^\infty_\delta(V, i\mathfrak{su}(E,H_0))$ satisfies

$$\mathfrak{L}(-\kappa,1) = \mathrm{Ad}(e^{-\kappa/2})(K_{H_{-1}}) - \kappa = 0. \qquad \square$$

4 Linearising $\mathfrak{L} = 0$

Having just proved that $1 \in I$, the next step is to show that $I \cap (0,1]$ is open. This will be established in this section by linearising the equation $\mathfrak{L} = 0$.

Since

$$\mathfrak{L}(s,t) = \mathrm{Ad}(e^{s/2})\left(K_{H_0} + i\Lambda\bar\partial(e^{-s}\partial_{H_0}e^s)\right) + t \cdot s,$$

it extends to a smooth map

$$\mathfrak{L}\colon\ C^{2,\alpha}_\delta(V, i\mathfrak{su}(E,H_0)) \times [0,1] \to C^{0,\alpha}_\delta(V, i\mathfrak{su}(E,H_0)).$$

The fact that $I \cap (0,1]$ is open is an immediate consequence of the following two propositions and the implicit function theorem for Banach spaces; see, e.g., [MS12, Theorem A.3.3].

Proposition 4.1. *If $(s,t) \in C^{2,\alpha}_\delta(V, i\mathfrak{su}(E,H_0)) \times (0,1]$ is a solution of $\mathfrak{L}(s,t) = 0$, then the linearisation*

$$L_{s,t} := \frac{\mathrm{d}\mathfrak{L}}{\mathrm{d}s}(s,t)\colon\ C^{2,\alpha}_\delta(V, i\mathfrak{su}(E,H_0)) \to C^{0,\alpha}_\delta(V, i\mathfrak{su}(E,H_0))$$

is invertible.

Proposition 4.2. *If $(s,t) \in C^{2,\alpha}_\delta(V, i\mathfrak{su}(E,H_0)) \times [0,1]$ is a solution of $\mathfrak{L}(s,t) = 0$, then*

$$s \in C^\infty_\delta(V, i\mathfrak{su}(E,H_0)).$$

The proofs of both of these results are essentially identical to those of the analogous results in the compact setting; see [LT06, Lemma 4.6 and Lemma 4.8]. The proofs make use of the explicit formulae for $\mathrm{Ad}(e^{s/2})K_{H_0e^s}$ and its derivative in the direction of s. The derivation of these, while rather straight-forward, is somewhat tedious and therefore relegated to Appendix A.

Proof of Proposition 4.2. By Proposition A.1 and since $\Theta(s)$ as defined in (A.3) is invertible, the equation $\mathfrak{L}(s,t)=0$ is equivalent to

$$\left(\frac{1}{2}\nabla^*_{H_0}\nabla_{H_0}+t\right)s+B(\nabla_{H_0}s\otimes\nabla_{H_0}s)=C(K_{H_0}).$$

where B and C are linear with coefficients depending on s, but not on its derivatives. The result now follows from a standard elliptic bootstrapping procedure. $\square$

Proof of Proposition 4.1. By Proposition A.8 and using

$$\mathfrak{L}(s,t)=\mathrm{Ad}(e^{s/2})K_{H_0e^s}-t\cdot s=0,$$

the linear operator $L_{s,t}$ is given by

$$L_{s,t}\hat{s}=\frac{1}{4}\nabla^*_{\tilde{A}_s}\nabla_{\tilde{A}_s}(\mathrm{id}+\mathrm{Ad}(e^{-s/2}))\Upsilon(s/2)\hat{s}+\frac{t}{4}\Big[s,(\mathrm{id}-\mathrm{Ad}(e^{-s/2}))\Upsilon(s/2)\hat{s}\Big]+t\hat{s}$$

with Υ as defined in (A.2). Since $s\in C^{2,\alpha}_\delta(V,i\mathfrak{su}(E,H_0))$, the linear operator $L_{s,t}$ can be connected to $\frac{1}{2}\nabla^*_{H_0}\nabla_{H_0}+t$ by a path of bounded linear operators which are asymptotic to $\frac{1}{2}(\partial^2_\ell-\partial^2_\theta+\nabla^*_{H_D}\nabla_{H_D})+t$. The argument in the proof of Proposition 2.9 shows that this is a path of Fredholm operators. Therefore, the index of $L_{s,t}$ agrees with that of $\frac{1}{2}\nabla^*_{H_0}\nabla_{H_0}+t$ and thus vanishes. To see that $L_{s,t}$ has trivial kernel and thus is invertible, observe that

$$\begin{aligned}\int_V&\langle L_{s,t}\hat{s},(\mathrm{id}+\mathrm{Ad}(e^{-s/2}))\Upsilon(s/2)\hat{s}\rangle\\&\geqslant t\int_V\langle(\mathrm{ad}_{s/4}(\mathrm{id}-\mathrm{Ad}(e^{-s/2})\Upsilon(s/2)+\mathrm{id})\hat{s},(\mathrm{id}+\mathrm{Ad}(e^{-s/2}))\Upsilon(s/2)\hat{s}\rangle\\&=t\int_V\langle\Xi(s)\hat{s},\hat{s}\rangle\end{aligned}$$

with

$$\Xi(s):=\Upsilon(s/2)(\mathrm{id}+\mathrm{Ad}(e^{-s/2}))(\mathrm{ad}_{s/4}(\mathrm{id}-\mathrm{Ad}(e^{-s/2})\Upsilon(s/2)+\mathrm{id}).$$

Since $\mathrm{Ad}(e^s)=e^{\mathrm{ad}_s}$ and $\mathrm{spec}(\mathrm{ad}_s)\subset\mathbf{R}$, it follows from

$$\frac{(e^{x/2}-1)}{x/2}(1+e^{-x/2})\left(\frac{x}{4}(1-e^{-x/2})\frac{e^{x/2}-1}{x/2}+1\right)=\frac{2\sinh(x)}{x}\geqslant 2,$$

for all $x\in\mathbf{R}$, that

$$\int_V\langle L_{s,t}\hat{s},\mathrm{Ad}(e^{-s/2})\Upsilon(s/2)\hat{s}\rangle\geqslant 2t\int_V|\hat{s}|^2.$$

Therefore, $L_{s,t}$ has trivial kernel. $\square$

5 A priori estimate

Given the following a priori estimate, it is an immediate consequence of Arzelà–Ascoli theorem that I is closed.

Proposition 5.1. *If $(s,t) \in C^\infty_\delta(V, i\mathfrak{su}(E,H_0)) \times [0,1]$ satisfies $\mathfrak{L}(s,t) = 0$, then*

$$\|s\|_{C^{k,\alpha}_\delta} \leqslant c_{k,\alpha}.$$

The proof of this proposition, to which this section is devoted, has two steps: First we prove that $\|s\|_{C^0}$ is bounded by a constant depending only on H_0 using ideas from [Sá 15]. This implies that $\|s\|_{C^k}$ is bounded by a constant depending only on k and H_0 by an argument of Bando and Siu [BS94, Proposition 1]. (For the reader's convenience we give a detailed proof of this in Appendix C.) The second step is a decay estimate which is similar to [HHN15, Steps 3 and 4 in the proof of Theorem 4.1].

5.1 A priori C^k estimate

Proposition 5.2. *If $(s,t) \in C^\infty_\delta(V, i\mathfrak{su}(E,H_0)) \times [0,1]$ satisfies $\mathfrak{L}(s,t) = 0$, then*

$$\|s\|_{C^k} \leqslant c_k.$$

Proof. By Theorem C.1 it suffices to prove the proposition for $k = 0$. Fix $L_0 \gg 1$, but independent of s, and set

$$N \coloneqq \|s\|_{L^\infty(V)} \quad \text{and} \quad M \coloneqq \|s\|_{L^\infty(V\setminus V_{L_0})}.$$

Step 1. *We have*

$$N \leqslant M + c(L_0 + 1).$$

We can assume that $|s|$ achieves its maximum at $x_0 \in V_{L_0}$ because otherwise the estimate holds trivially. From Proposition A.9 and $\mathfrak{L}(s,t) = 0$ it follows that

$$\Delta|s|^2 + 4t|s|^2 \leqslant -4\langle K_{H_0}, s\rangle; \tag{5.3}$$

hence,

$$\Delta|s|^2 \leqslant 4N|K_{H_0}|.$$

Denote by $f \in C^{2,\alpha}_\delta(V)$ and $A > 0$ the unique solution to

$$\Delta(f - A\ell) = 4|K_{H_0}|.$$

Applying the maximum principle to the subharmonic function $|s|^2 - N(f - A\ell)$ on V_{L_0} we have

$$N^2 \leqslant M^2 + N(AL_0 + 2\|f\|_{L^\infty}) \leqslant N(M + AL_0 + 2\|f\|_{L^\infty}).$$

This implies the assertion.

Step 2. *We have*

$$\sqrt{M} \lesssim \|K_{H_0 e^s|_{D_z}}\|_{L^2(V \setminus V_{L_0})} + 1.$$

Here D_z is as in (2.2) for $z = (L, \theta) \in (L_0, \infty) \times S^1$.

Step 2.1. *If $x_0 \in \overline{V \setminus V_{L_0}}$ is such that*

$$|s|(x_0) = M,$$

then for all $L \geqslant \ell(x_0)$ we have

$$\|s\|_{L^\infty(\partial V_L)} - \frac{1}{4}M \gtrsim \ell(x_0) - L.$$

By the maximum principle applied to $|s|^2 - N(f - A\ell)$ on V_L we have

$$M^2 - Nf(x_0) + NA\ell(x_0) \leqslant \|s\|^2_{L^\infty(\partial V_L)} + N\|f\|_{L^\infty(\partial V_L)} + NAL.$$

We can assume that $M \geqslant 8\|f\|_{L^\infty(V \setminus V_{L_0})}$ and $N \leqslant 2M$, because otherwise N can already be bounded independent of s using Step 1. With these assumptions it follows that

$$\begin{aligned} NA(\ell(x_0) - L) &\leqslant \|s\|^2_{L^\infty(\partial V_L)} - M^2 + 2N\|f\|_{L^\infty(V \setminus V_{L_0})} \\ &\leqslant N\left(\|s\|_{L^\infty(\partial V_L)} - \frac{1}{4}M\right). \end{aligned}$$

Step 2.2. *There are $L_0 \leqslant L_1 < L_2$ with $L_2 - L_1 \asymp M$ such that*

$$M^{3/2} \lesssim \|s\|_{L^2(V_{L_2} \setminus V_{L_1})}.$$

By Step 2.1 we have

$$M \lesssim \|s\|_{L^\infty(\partial V_L)}$$

for $0 \leqslant L - \ell(x_0) \asymp M$; hence, using the mean value inequality [GT01, Theorem 9.20], $\Delta|s|^2 \leqslant 4|K_{H_0}||s|$, and $|K_{H_0}| = e^{-\delta\ell}$ it follows that

$$M^2 \lesssim \int_{V_{L+1} \setminus V_{L-1}} |s|^2 + e^{-\delta L_0} M.$$

Since $L_0 \gg 1$, the second term on the right-hand side can be rearranged. Summing over $L - \ell(x_0) = 1, \ldots, k$ (with $k \asymp M$) yields the asserted inequality.

Step 2.3. *We have*

$$\|s\|_{L^2(D_z)} - 1/2 \lesssim M\|K_{H_0 e^s|_{D_z}}\|_{L^2(D_z)}.$$

At this stage the μ–stability of $\mathscr{E}_D$ comes into play via the Donaldson functional $\mathscr{M}$; see Appendix B. Since $L_0 \gg 1$ and $\mathscr{E}_D$ is μ–stable, $\mathscr{E}|_{D_z}$ is μ–stable as well. Denote by H_{D_z} the **P**HYM metric on $\mathscr{E}_{D_z}$ inducing the same metric on $\det(\mathscr{E}|_{D_z})$ as $H_0|_{D_z}$. Set $\sigma_z := \log(H_{D_z}^{-1} H_0|_{D_z})$. By the construction of H_0 in Proposition 3.2 we have $\sigma_z \in C^\infty_\delta(V, i\mathfrak{su}(E, H_0))$.

Using Theorem B.3, Proposition B.1, and Proposition B.2 we have

$$\begin{aligned}
\|s|_{D_z}\|_{L^2(D_z)} - 1 &\lesssim \|\log(e^\sigma e^{s|_{D_z}})\|_{L^2(D_z)} - 1 \\
&\lesssim \mathscr{M}(H_{D_z}, H_{D_z} e^\sigma e^{s|_{D_z}}) \\
&\lesssim \mathscr{M}(H_0|_{D_z}, H_{D_z}) + \mathscr{M}(H_0|_{D_z}, H_0 e^s|_{D_z}) \\
&\lesssim \int_{D_z} |s||K_{H_0 e^s|_{D_z}}| + e^{-\delta L_0}.
\end{aligned}$$

This implies the asserted inequality.

Comparing the lower bounds from Step 2.2 with the upper bounds obtained by integrating Step 2.3 completes the proof of Step 2.

Step 3. *We have*

$$\|K_{H_0 e^s|_{D_z}}\|^2_{L^2(V\setminus V_{L_0})} \lesssim e^{-\delta L_0} + \|F^\circ_{H_0}\|^2_{L^2(V_{L_0})}.$$

Here $F^\circ_{H_0}$ denotes the curvature of the $\mathrm{PU}(r)$*–connection induced by H_0.*

Once this is proved, the desired control on M follows and the proof of Proposition 5.2 will be complete.

Step 3.1. *We have*

$$\|K_{H_0 e^s|_{D_z}}\|^2_{L^2(V\setminus V_{L_0})} \lesssim \int_V |F^\circ_{H_0 e^s}|^2 - |F^\circ_{H_0}|^2 + ce^{-\delta L_0} + \|F^\circ_{H_0}\|^2_{L^2(V_{L_0})}.$$

If H is a Hermitian metric on a holomorphic bundle $\mathscr{E}$ over an n–dimensional Kähler manifold X with Kähler form ω, then

$$q_4(H) \wedge \omega^{n-2} = c\left(|F^\circ_H|^2 - |K_H|^2\right) \mathrm{vol} \tag{5.4}$$

with

$$q_4(H) := 2c_2(H) - \frac{r-1}{r} c_1(H)^2$$

and c_k denoting the k–th Chern form associated with H.

If X is compact, then the integral of the left-hand side of (5.4) depends only $\mathscr{E}$; hence,

$$\int_{D_z} |K_{H_0 e^s|_{D_z}}|^2 = \int_{D_z} |K_{H_0|_{D_z}}|^2 + \int_{D_z} |F^\circ_{H_0 e^s|_{D_z}}|^2 - |F^\circ_{H_0|_{D_z}}|^2.$$

Since

$$|F_{H_0} - F_{H_0|_{D_z}}| \lesssim e^{-\delta L} \quad \text{and} \quad |K_{H_0|_{D_z}}| \lesssim e^{-\delta L},$$

it follows that

$$\begin{aligned}\int_{D_z} |K_{H_0 e^s|_{D_z}}|^2 &\lesssim \int_{D_z} |F^\circ_{H_0 e^s|_{D_z}}|^2 - |F^\circ_{H_0|_{D_z}}|^2 + e^{-\delta L} \\ &\lesssim \int_{D_z} |F^\circ_{H_0 e^s}|^2 - |F^\circ_{H_0}|^2 + e^{-\delta L}.\end{aligned}$$

Step 3.2. *We have*

$$\int_V |F^\circ_{H_0 e^s}|^2 - |F^\circ_{H_0}|^2 \leqslant 0.$$

Since $s \in C^\infty_\delta(V, i\mathfrak{su}(E, H_0))$, we have

$$\int_V (q_4(H_0 e^s) - q_4(H_0)) \wedge \omega^{n-2} = 0.$$

Using (5.4), we obtain

$$\int_V |F^\circ_{H_0 e^s}|^2 - |F^\circ_{H_0}|^2 = \int_V |K_{H_0 e^s}|^2 - |K_{H_0}|^2.$$

To see that the right-hand side is non-positive, we use (5.3) and $\mathfrak{L}(s, t) = 0$ to derive

$$\begin{aligned}\int_V |K_{H_0 e^s}|^2 = \int_V t^2 |s|^2 &\leqslant \int_V t |K_{H_0}||s| \\ &\leqslant \int_V \frac{1}{2}|K_{H_0}|^2 + \frac{1}{2} t^2 |s|^2 = \int_V \frac{1}{2}|K_{H_0}|^2 + \frac{1}{2}|K_{H_0 e^s}|^2.\end{aligned}$$ □

5.2 Decay estimate

Proof of Proposition 5.1. To complete the proof we need to establish quantitative exponential decay bounds for s using the a priori estimate in Proposition 5.2 and the qualitative information that $s \in C^\infty_\delta(V, i\mathfrak{su}(E, H_0))$.

Fix $L_0 \gg 1$ as in the proof of Proposition 5.2.

Step 1. *We have*

$$\int_{V \setminus V_{L_0}} |\nabla_{H_0} s|^2 \leqslant c.$$

From Proposition A.9 and $\mathfrak{L}(s, t) = 0$ it follows that

$$\Delta |s|^2 + 2|\upsilon(-s)\nabla_{H_0} s|^2 \leqslant -4\langle K_{H_0}, s\rangle$$

with $\upsilon(-s)$ as defined in (A.11). Since

$$\upsilon(-s) = \sqrt{\frac{1 - e^{-\operatorname{ad}_s}}{\operatorname{ad}_s}} \quad \text{and} \quad \sqrt{\frac{1 - e^{-x}}{x}} \gtrsim \frac{1}{\sqrt{1 + |x|}},$$

it follows that

$$|\nabla_{H_0} s|^2 \lesssim (1 + \|s\|_{L^\infty}) \left(|K_{H_0}||s| - \Delta|s|^2\right). \tag{5.5}$$

Integrating this over V and using (3.1) as well as Proposition 5.2 yields the asserted estimate.

Step 2. *For some $\varepsilon > 0$ and all $L \geqslant L_0$, we have*

$$\int_{V\setminus V_L} |s|^2 \lesssim e^{-2\varepsilon L} \quad \textit{and} \quad \int_{V\setminus V_L} |\nabla_{H_0} s|^2 \lesssim e^{-2\varepsilon L}.$$

Since $\mathscr{E}_D$ is simple, for all $\tilde{s} \in \Gamma(D, \mathscr{E}\mathrm{nd}_0(\mathscr{E}_D))$ we have

$$\int_D |\tilde{s}|^2 \lesssim \int_D |\bar{\partial}_D \tilde{s}|^2 \lesssim \int_D |\nabla_{H_D} \tilde{s}|^2.$$

Because $L_0 \gg 1$, this implies that

$$\int_{\partial V_L} |s|^2 \lesssim \int_{\partial V_L} |\nabla_{H_0} s|^2 \tag{5.6}$$

for $L \geqslant L_0$. Therefore, it suffices to prove the second inequality.

Integrating (5.5) over $V\setminus V_L$ and using (5.6) yields

$$\begin{aligned} \int_{V\setminus V_L} |\nabla_{H_0} s|^2 &\lesssim e^{-\delta L} + \int_{\partial V_L} |\nabla_{H_0} s||s| \\ &\lesssim e^{-\delta L} + \int_{\partial V_L} |\nabla_{H_0} s|^2. \end{aligned}$$

The assertion now follows from Proposition 5.8, which will be proved at the end of this section.

Step 3. *With $\varepsilon > 0$ as above*

$$\|s\|_{C^{k,\alpha}_\varepsilon} \leqslant c_{k,\alpha}.$$

As in the proof of Proposition 4.2, we can write $\mathfrak{L}(s, t) = 0$ in the form

$$\left(\frac{1}{2}\nabla^*_{H_0}\nabla_{H_0} + t\right) s + B(\nabla_{H_0} s \otimes \nabla_{H_0} s) = \mathfrak{e}, \tag{5.7}$$

where B is linear with coefficients depending on s, and by (3.1)

$$\|\mathfrak{e}\|_{C^{k,\alpha}_\delta} \leqslant c_{k,\alpha}.$$

Using standard interior estimates the assertion follows from Proposition 5.2 and Step 2.

Step 4. *We prove the proposition.*

Since

$$\|\nabla_{H_0} s \otimes \nabla_{H_0} s\|_{C^{k,\alpha}_{2\varepsilon}} \lesssim \|\nabla_{H_0} s\|^2_{C^{k,\alpha}_{\varepsilon}},$$

we note that

$$\left\|\frac{1}{2}\nabla^*_{H_0}\nabla_{H_0} s + ts\right\|_{C^{k,\alpha}_{\varepsilon'}} \leqslant c_{k,\alpha}$$

with $\varepsilon' := \min\{2\varepsilon, \delta\}$. From Proposition 2.9 it follows that

$$\|s\|_{C^{k,\alpha}_{\varepsilon'}} \leqslant c_{k,\alpha}.$$

Repeating this argument a finite number of times we finally arrive at $\varepsilon' = \delta$. □

Proposition 5.8. *If $f\colon [0,\infty) \to [0,\infty)$ satisfies*

$$f(L) \leqslant Ae^{-\delta L} - Bf'(L)$$

with $A, B > 0$, then

$$f(L) \leqslant (2A + f(0))e^{-\varepsilon L}$$

with $\varepsilon := \min\{\delta, 1/2B\}$.

Proof. The function $g\colon [0,\infty) \to \mathbf{R}$ defined by

$$g(L) := f(L) - (2A + f(0))e^{-\varepsilon L}$$

satisfies $g(0) = -2A \leqslant 0$ and $g'(L) \leqslant -g(L)/B$. It follows that $g \leqslant 0$, which proves the proposition. □

6 The Bando–Siu continuity method

To prove Theorem 1.1 for reflexive sheaves $\mathscr{E}$ we use a regularization scheme based on ideas of Bando and Siu [BS94]. We construct a one-parameter family of ACyl Kähler manifolds $\{\tilde{V}_\varepsilon : \varepsilon \in (0,1]\}$ whose underlying complex manifold $\tilde{V}$ is obtained by blowing up $S := \operatorname{sing}(\mathscr{E})$. As ε tends to zero, the exceptional divisor shrinks and $\tilde{V}_\varepsilon$ resembles V more and more closely. $\tilde{V}$ carries a holomorphic vector bundle $\tilde{\mathscr{E}}$, which agrees with $\mathscr{E}$ outside S, and to which Theorem 1.1 can be applied to construct a **P**HYM metric $\tilde{H}_\varepsilon$. The desired **P**HYM metric on $\mathscr{E}$ will be constructed by taking the limit as ε tends to zero.

Proposition 6.1. *There is a complex manifold $\tilde{V}$, a holomorphic map $\hat{\pi}\colon \tilde{V} \to V$ which induces a biholomorphic map to $V\backslash S$, and a holomorphic vector bundle $\tilde{\mathscr{E}}$ over $\tilde{V}$ such that*

$$\tilde{\mathscr{E}}|_{\tilde{V}\backslash\hat{\pi}^{-1}(S)} \cong \hat{\pi}^*(\mathscr{E}|_{V\backslash S}).$$

Moreover, there exists a one-parameter family of Kähler metrics $\{g_\varepsilon : \varepsilon \in (0,1]\}$ on $\tilde{V}$ such that:

- *on $\hat{\pi}^{-1}(V \backslash B_{\sqrt{\varepsilon}}(S))$ we have $g_\varepsilon = \hat{\pi}^* g$, and*
- *for $L \geqslant L_0$, the Neumann–Poincaré constant of $(\hat{\pi}^{-1}(V_L), g_\varepsilon)$ is bounded above by a constant independent of ε. Here L_0 is as in Definition 2.3.*

Proof. The proof has three steps.

Step 1. *Construction of $\tilde{V}$ and $\tilde{\mathscr{E}}$.*

We follow the method of Bando and Siu [BS94, p. 46], see also [Sib15, Section 4.1].

Since $\mathscr{E}^*$ is coherent, there exists a locally free sheaf $\mathscr{F}$ and a surjective morphism $\mathscr{F}^* \to \mathscr{E}^* \to 0$. Since $\mathscr{E}$ is reflexive, by dualising, we get $0 \to \mathscr{E} \to \mathscr{F}$. This defines a rational section $\phi_{\mathscr{E}} \colon V \dashrightarrow \mathrm{Gr}_r(\mathscr{F})$, with locus of indeterminacy S. By a result of Hironaka [Hir64, Part I, Chapter 0, Section 5], there exists a holomorphic map $\hat{\pi} \colon \tilde{V} \to V$, which is biholomorphic outside S and equivalent to a sequence of blow-ups along smooth submanifolds (of codimension at least three), such that $\phi_{\mathscr{E}} \circ \hat{\pi}$ extends to a section $\tilde{V} \to \mathrm{Gr}_r(\hat{\pi}^*\mathscr{F})$. This section defines the desired holomorphic vector bundle $\tilde{\mathscr{E}}$ over $\tilde{V}$.

Step 2. *The model metric.*

Denote by ω_{FS} the Fubini–Study form on $\mathbf{P}^{n-1}$. The Kähler form

$$\tilde{\omega}_\varepsilon = i\partial\bar{\partial}\left(\frac{1}{2}|z|^2 + \frac{\varepsilon^2}{2\pi}\log|z|^2\right)$$

on $\mathbf{C}^n \backslash \{0\}$ uniquely extends to a Kähler form on $\mathrm{Bl}_0\,\mathbf{C}^n$ which induces $\varepsilon^2\omega_{FS}$ on the exceptional divisor $\mathbf{P}^{n-1}$. More precisely, if we denote by r the radial coordinate, by θ the 1–form arising from the S^1–action and by $\varpi \colon \mathbf{C}^n \backslash \{0\} \to \mathbf{P}^{n-1}$ the projection, then

$$\tilde{\omega}_\varepsilon = (\varepsilon^2 + r^2)\varpi^*\omega_{FS} + r\mathrm{d}r \wedge \theta. \tag{6.2}$$

Fix a smooth function $\chi \colon [0, \infty) \to [0, 1]$ which is equal to one on $[0, 1]$ and vanishes outside $[0, 2]$. For $0 < \varepsilon \ll 1$, set $\chi_\varepsilon \coloneqq \chi(\cdot/2\sqrt{\varepsilon})$ and define a Kähler form on $\mathrm{Bl}_0\,\mathbf{C}^n$ by

$$\omega_\varepsilon \coloneqq i\partial\bar{\partial}\left(\frac{1}{2}|z|^2 + \chi_\varepsilon(|z|) \cdot \frac{\varepsilon^2}{2\pi}\log|z|^2\right).$$

This agrees with $\tilde{\omega}_\varepsilon$ on $B_{\sqrt{\varepsilon}/2}$, it agrees with the flat Kähler form ω_0 on $\mathbf{C}^n \backslash B_{\sqrt{\varepsilon}}(0)$, and it satisfies

$$|\omega_\varepsilon - \omega_0| \lesssim \varepsilon|\log\varepsilon|$$

on $B_{\sqrt{\varepsilon}}(0) \backslash B_{\sqrt{\varepsilon}/2}(0)$. Moreover, we have

$$\frac{\omega_\varepsilon^n}{\omega^n} \asymp 1 + (\varepsilon/r)^{2n-2}.$$

Step 3. *Construction of g_ε.*

$\tilde{V}$ is constructed by a finite sequence of blow-ups along smooth submanifolds. In fact, by induction we can assume that there is just one blow-up, say, along $C \subset V$. Denote by $\rho\colon V \longrightarrow [0,\infty)$ the distance to C. For $0 < \varepsilon \leqslant \varepsilon_0$,

$$\omega_\varepsilon \coloneqq \hat{\pi}^*\omega + i\partial\bar{\partial}\left((\chi_\varepsilon \circ \rho) \cdot \frac{\varepsilon^2}{2\pi}\log\rho^2\right)$$

defines a Kähler form on $\tilde{V}$ whose restriction to $\hat{\pi}^{-1}(V\backslash B_\varepsilon(S))$ agrees with $\hat{\pi}^*\omega$. We extend the resulting family of Kähler metrics to be constant for $\varepsilon \in [\varepsilon_0, 1]$.

Step 4. *Estimate of the Neumann–Poincaré constant.*

Fix $L \geqslant L_0$. We use the discretization method of Grigor'yan and Saloff-Coste [GS05, Section 3.1] to estimate the Neumann–Poincaré constant of $(\hat{\pi}^{-1}(V_L), g_\varepsilon)$. Fix $0 < \sigma \ll 1$. Pick a maximal set of points $\{x_j : j \in J\} \subset V_{L-1/2}$ of distance at least σ from each other. Set

$$A_0 \coloneqq V_L\backslash V_{L-1/2}, \quad A_0^* = A_0^\# \coloneqq V_L\backslash V_{L-1},$$
$$A_j \coloneqq \hat{\pi}^{-1}(B_\sigma(x_j)), \quad A_j^* = \hat{\pi}^{-1}(B_{4\sigma}(x_j)) \quad \text{and} \quad A_j^\# \coloneqq \hat{\pi}^{-1}(B_{8\sigma}(x_j)).$$

Set $I \coloneqq J \sqcup \{0\}$. $\mathscr{A} \coloneqq \left\{(A_i, A_i^*, A_i^\#) : i \in I\right\}$ is a **good covering** of V_L in V_L in the sense of Grigor'yan and Saloff-Coste [GS05, Definition 3.1]. This means that, for all $i \in I$, $A_i \subset A_i^* \subset A_i^\#$ and for some constants Q_1, Q_2 the following hold:

- We have $V_L \subset \bigcup_{i\in I} A_i$ and $\bigcup_{i\in I} A_i^\# \subset V_L$.
- For each $i \in I$, $|\{j \in I : A_i^\# \cap A_j^\# \neq \varnothing\}| \leqslant Q_1$.
- If $d(A_i, A_j) = 0$, then there is a $k = k(i,j) \in I$ such that $A_i \cup A_j \subset A_k^*$. Moreover, $\mathrm{vol}(A_k^*) \leqslant Q_2 \min\{\mathrm{vol}(A_i), \mathrm{vol}(A_j)\}$.

According to [GS05, Theorem 3.7] the Neumann–Poincaré constant of V_L can be estimated by $Q_1\Lambda_c(2 + Q_1^2 Q_2 \Lambda_d)$. Here the **continuous Poincaré constant** Λ_c and the **discrete Poincaré constant** Λ_d [GS05, Definition 3.4 and Definition 3.6] are the smallest constants such that,

$$\int_{A_i} |f - \bar{f}_{A_i}|^2 \leqslant \Lambda_c \int_{A_i^*} |\nabla f|^2 \quad \text{and} \quad \int_{A_i^*} |f - \bar{f}_{A_i^*}|^2 \leqslant \Lambda_c \int_{A_i^\#} |\nabla f|^2 \tag{6.3}$$

and

$$\sum_{i\in I} |f(i) - \bar{f}|^2 m(i) \leqslant \Lambda_d \mathscr{E}(f,f).$$

Here

$$m(i) = \mathrm{vol}(A_i), \quad \bar{f} \coloneqq \frac{\sum_{i\in I} f(i)m(i)}{\sum_{i\in I} m(i)} \quad \text{and}$$
$$\mathscr{E}(f,f) \coloneqq \frac{1}{2}\sum_{(i,j)\in I\times I} |f(i) - f(j)|^2 m(i,j).$$

with

$$m(i,j) := \begin{cases} \max\{m(i), m(j)\} & \text{if } d(A_i, A_j) = 0 \\ 0 & \text{otherwise.} \end{cases}$$

While the measures of A_i, A_i^*, and $A_i^\#$ are dependent of ε, they are uniformly comparable. Consequently, the constants Q_1 and Q_2 and discrete Poincaré constant Λ_d can be bounded independently of ε. Thus it remains to show that Λ_c can be bounded independently of ε; that is, we can find a constant such that (6.3) holds for all $i \in I$ and $\varepsilon \in (0,1]$. For $i = 0$, (6.3) is obvious. For $i \in J$, such estimates follow from scaling considerations and uniform weak Poincaré inequalities

$$\int_{B_r(x)} |f - \bar{f}_{B_r(x)}|^2 \leqslant cr^2 \int_{B_{2r}(x)} |\nabla f|^2$$

(with $c > 0$ independent of x and r) for certain model spaces, for example, $\mathrm{Bl}_0\, \mathbf{C}^k \times \mathbf{C}^{n-k}$ equipped with the Kähler metric induced by $i\partial\bar{\partial}\left(\frac{1}{2}|z|^2 + \frac{1}{2\pi}\log|z|^2 + \frac{1}{2}|w|^2\right)$. The existence of these uniform Poincaré constants in turn can also be established using the discretization method as follows. We can assume that $r \gg 1$. Denote by $\hat{\pi}\colon\ \mathrm{Bl}_0\, \mathbf{C}^k \times \mathbf{C}^{n-k} \to \mathbf{C}^n$ the projection. For $i \in \mathbf{Z}^{2n} \subset \mathbf{C}^n$, set

$$A_i := \hat{\pi}^{-1}(B_1(i)), \quad A_i^* := \hat{\pi}^{-1}(B_4(i)) \quad \text{and} \quad A_i^\# := \hat{\pi}^{-1}(B_8(i)).$$

If we set $I_{x,r} := \{i \in \mathbf{Z}^{2n} \cap \hat{\pi}(B_r(x))\}$, then $\mathscr{A}_{x,r} := \{(A_i, A_i^*, A_i^\#) : i \in I_{x,r}\}$ is a good covering of $B_r(x)$ in $B_{2r}(x)$; moreover, the constants Q_1 and Q_2 as well as the continuous Poincaré constant Λ_c of $\mathscr{A}_{x,r}$ can be bounded independently of x and r. The discrete Poincaré constant of $\mathscr{A}_{x,r}$ can be bounded by a constant times r^2; see, e.g., [Ber14, Section 3.4]. [GS05, Theorem 3.7] thus establishes the desired uniform weak Poincaré inequalities. □

We denote $\tilde{V}$ equipped with the metric g_ε by $\tilde{V}_\varepsilon$. Given a subset $U \subset V$, we set $\tilde{U} := \hat{\pi}^{-1}(U)$.

Using Theorem 1.1 for holomorphic vector bundles, for each $\varepsilon \in (0,1]$, we construct a **P**HYM metric $\tilde{H}_\varepsilon$ on $\tilde{\mathscr{E}}$ over $\tilde{V}_\varepsilon$. We can assume that the metric on $\det\tilde{\mathscr{E}}$ induced by $\tilde{H}_\varepsilon$ agrees with a fixed asymptotically translation-invariant metric $\tilde{h}$ which does not depend on ε. Define $\tilde{s}_\varepsilon \in C^\infty_\delta(\tilde{V}_\varepsilon, i\mathfrak{su}(E, \tilde{H}_1))$ by

$$\tilde{s}_\varepsilon := \log \tilde{H}_1^{-1}\tilde{H}_\varepsilon.$$

The **P**HYM metric H on $\mathscr{E}$, whose existence was asserted in Theorem 1.1, can be constructed using the following proposition and the Arzelà–Ascoli theorem by taking the limit of the metrics $\tilde{H}_\varepsilon$ over $V\backslash U = \tilde{V}_\varepsilon\backslash\tilde{U}$ as ε tends to zero. Here U is an arbitrary neighbourhood of $S \subset V$.

Proposition 6.4. *For all $\varepsilon \in (0,1]$, we have*

$$\|\tilde{s}_\varepsilon\|_{C^k_\delta(\tilde{V}_\varepsilon\backslash\tilde{U})} \leqslant c_{k,U}.$$

Proof. Set

$$K_\varepsilon := i\Lambda_\varepsilon F_{\tilde{H}_1} - \frac{\mathrm{tr}(i\Lambda_\varepsilon F_{\tilde{H}_1})}{\mathrm{rk}\,\tilde{\mathscr{E}}} \cdot \mathrm{id}_{\tilde{\mathscr{E}}},$$

and let $f_\varepsilon \in C^0_\delta(\tilde V_\varepsilon)$ and $A_\varepsilon > 0$ be the unique solution to

$$\Delta_\varepsilon(f_\varepsilon - A_\varepsilon \ell) = 4|K_\varepsilon|.$$

Here Λ_ε and Δ_ε denote the dual Lefschetz operator and the Laplace operator on $\tilde V_\varepsilon$ respectively.

If we can prove that

$$\|f_\varepsilon\|_{L^\infty(\tilde V_\varepsilon \setminus \tilde U)} \leqslant c_U, \quad A_\varepsilon \leqslant c \quad \text{and} \quad \|F_{\tilde H_1}\|_{L^2(\tilde V_{\varepsilon,L_0})} \leqslant c,$$

then the argument in Section 5 will yield the asserted bounds on $\tilde s_\varepsilon$.

The proof of the above bounds on f_ε, A_ε and $F_{\tilde H_1}$ proceeds in four steps.

Step 1. *We have*

$$\|F_{\tilde H_1}\|_{L^2(\tilde V_{\varepsilon,L_0})} \leqslant c \quad \textit{and} \quad \|K_\varepsilon\|_{C^k_\delta(V\setminus V_{L_0})} \leqslant c_k;$$

in particular, $A_\varepsilon \leqslant c$.

Recall that ρ denotes the distance to S. By (6.2), we have

$$\frac{\mathrm{vol}_{g_\varepsilon}}{\mathrm{vol}_{g_\varepsilon}} \lesssim \left(\frac{\rho^2+\varepsilon^2}{\rho^2+1}\right)^{\mathrm{codim}(S)-1} \quad \text{and} \quad \frac{|\beta|_{g_\varepsilon}}{|\beta|_{g_1}} \lesssim \left(\frac{\rho^2+\varepsilon^2}{\rho^2+1}\right)^{-1}$$

for any 2–form β. Consequently,

$$|F_{\tilde H_1}|^2_{g_\varepsilon}\mathrm{vol}_{g_\varepsilon} \lesssim \left(\frac{\rho^2+\varepsilon^2}{\rho^2+1}\right)^{\mathrm{codim}(S)-3} |F_{\tilde H_1}|^2\mathrm{vol}_{g_1}.$$

Since $\operatorname{codim} S \geqslant 3$, this implies the asserted L^2–bound. The second inequality is a consequence of the fact that g_ε, and thus K_ε, does not depend on ε on $V\setminus V_{L_0}$. Both estimates together yield $A_\varepsilon \lesssim \|K_\varepsilon\|_{L^1(\tilde V_\varepsilon)} \leqslant c$.

Step 2. *There is a constant* $\bar f_\varepsilon$ *such that on* $V\setminus V_{L_0}$ *we have*

$$\|e^{-\frac{\delta\ell}{2}}(f_\varepsilon - \bar f_\varepsilon)\|_{L^2(\tilde V_\varepsilon)} \leqslant c \quad \textit{and} \quad \|\nabla_\varepsilon f_\varepsilon\|^2_{L^2(\tilde V_\varepsilon)} \leqslant c.$$

From Proposition 6.1 it follows that the weighted Neumann–Poincaré inequality [HHN15, Theorem 4.18] holds for $\sigma = 1$ and $\mu = \frac{\delta}{2}$ with a constant $c > 0$ independent of ε; hence, for some constant $\bar f_\varepsilon$

$$\|e^{-\frac{\delta\ell}{2}}(f_\varepsilon - \bar f_\varepsilon)\|^2_{L^2(\tilde V_\varepsilon)} \lesssim \|\nabla_\varepsilon f_\varepsilon\|^2_{L^2(\tilde V_\varepsilon)}.$$

Using the previous step, we have

$$\begin{aligned}
\|\nabla_\varepsilon f_\varepsilon\|^2_{L^2(\tilde V_\varepsilon)} &= \int_{\tilde V_\varepsilon} \langle \Delta_\varepsilon(f_\varepsilon - \bar f_\varepsilon), f_\varepsilon - \bar f_\varepsilon\rangle \\
&\leqslant \|e^{\frac{\delta\ell}{2}}(K_\varepsilon + A_\varepsilon\Delta_\varepsilon \ell)\|_{L^2(\tilde V_\varepsilon)} \cdot \|e^{-\frac{\delta\ell}{2}}(f_\varepsilon - \bar f_\varepsilon)\|_{L^2(\tilde V_\varepsilon)} \\
&\lesssim \|e^{-\frac{\delta\ell}{2}}(f_\varepsilon - \bar f_\varepsilon)\|_{L^2(\tilde V_\varepsilon)}.
\end{aligned}$$

Combined with the above this yields

$$\|e^{-\frac{\delta\ell}{2}}(f_\varepsilon - \bar{f}_\varepsilon)\|_{L^2(\tilde{V}_\varepsilon)} \leqslant c.$$

This in turn implies the second of the asserted inequalities.

Step 3. *We have*

$$\|f_\varepsilon\|_{L^\infty(\tilde{V}_\varepsilon\setminus U)} \leqslant c_U.$$

Define $F\colon [L_0, \infty) \to [0, \infty)$ by

$$F(L) \coloneqq \int_{V\setminus V_{L_0}} |\nabla_\varepsilon f_\varepsilon|^2.$$

By the previous step, we have

$$F(L) \leqslant c.$$

Setting $\bar{f}_{\varepsilon,L} \coloneqq \fint_{\partial V_L} f_\varepsilon$, we have

$$\int_{\partial V_L} |f_\varepsilon - \bar{f}_{\varepsilon,L}| \leqslant \int_{\partial V_L} |f_\varepsilon - \bar{f}_\varepsilon|.$$

Using integration by parts, the Neumann–Poincaré inequality on ∂V_L, and the previous step, we have

$$\begin{aligned} F(L) &\leqslant \int_{V\setminus V_L} |K_\varepsilon + A_\varepsilon \Delta\ell||f_\varepsilon - \bar{f}_{\varepsilon,L}| + \int_{\partial V_L} |\nabla_\varepsilon f||f_\varepsilon - \bar{f}_{\varepsilon,L}| \\ &\lesssim \int_{V\setminus V_L} e^{-\delta\ell}|f_\varepsilon - \bar{f}_\varepsilon| + \int_{\partial V_L} |\nabla_\varepsilon f||f_\varepsilon - \bar{f}_{\varepsilon,L}| \\ &\lesssim e^{-\frac{\delta L}{2}} - F'(L). \end{aligned}$$

It follows from Proposition 5.8 that $F(L) \lesssim e^{-2\gamma L}$ for some $\gamma > 0$. From interior estimates it follows that

$$|\nabla_\varepsilon f_\varepsilon| \lesssim e^{-\gamma\ell}$$

on $V\setminus V_{L_0}$ and

$$\|\nabla_\varepsilon f_\varepsilon\|_{L^\infty(\tilde{V}_\varepsilon\setminus U)} \leqslant c_U.$$

By the exponential decay of f_ε, the above bound implies the assertion by integrating the gradient of f_ε along a path down the the tubular end of V. ◻

The L^2 curvature bound asserted in Theorem 1.1 is a consequence of the following proposition.

Proposition 6.5. *For each $\varepsilon \in (0, 1]$, we have*

$$\left\|F_{\tilde{H}_\varepsilon}\right\|_{L^2(\tilde{V}_{\varepsilon,L})} \lesssim L + 1.$$

Proof. Since $\tilde{h}$ is fixed, it suffices to estimate $F^{\circ}_{\tilde{H}_\varepsilon}$, the curvature of the $\mathbf{P}\mathrm{U}(r)$–connection induced by $\tilde{H}_\varepsilon$.

For each fixed $\varepsilon \in (0,1]$, we have a bound of the desired form; however, it might a priori depend on ε. To see that it does not, we use a topological argument. With q_4 as defined in (5.4) we have

$$q_4(\tilde{H}_\varepsilon) - q_4(\tilde{H}_1) = \mathrm{d}\tau(\tilde{s}_\varepsilon)$$

where τ is the transgression form associated with q_4 and can be bounded in terms of $|\tilde{s}_\varepsilon|$ and $|\nabla_{\tilde{H}}\tilde{s}_\varepsilon|$. Using (5.4) and $K_{\tilde{H}_\varepsilon} = 0$, we derive

$$\begin{aligned}
\int_{\tilde{V}_L} \left|F^{\circ}_{\tilde{H}_\varepsilon}\right|^2 \mathrm{vol}_\varepsilon &\lesssim \int_{\tilde{V}_L} q_4(\tilde{H}_\varepsilon) \wedge \omega_\varepsilon^{n-2} \\
&= \int_{\tilde{V}_L} (q_4(\tilde{H}_1) + \mathrm{d}\tau) \wedge \omega_\varepsilon^{n-2} \\
&\lesssim \int_{\tilde{V}_L} \left|F^{\circ}_{\tilde{H}_1}\right|^2_{g_\varepsilon} \mathrm{vol}_\varepsilon + 1 \\
&\lesssim \int_{\tilde{V}_L} \left|F^{\circ}_{\tilde{H}_1}\right|^2_{g_1} \mathrm{vol}_1 + 1 \\
&\lesssim L + 1.
\end{aligned}$$

Here the second term in the third step arises from Stokes' theorem and the fourth step uses the argument from Step 1 in the proof of Proposition 6.4. ◻

This finishes the proof of Theorem 1.1.

7 Uniqueness of PHYM metrics

We have the following basic uniqueness result for asymptotically translation-invariant $\mathbf{P}$HYM metrics.

Proposition 7.1. *Let $\mathscr{E}$ be a reflexive sheaf over V asymptotic to $\mathscr{E}_D$ and let h be an asymptotically translation-invariant Hermitian metric on $\det E$. If $\mathscr{E}_D$ is simple, then there exists at most one asymptotically translation-invariant $\mathbf{P}$HYM metric on $\mathscr{E}$ inducing h.*

Proof. If H_0 and H were two asymptotically translation-invariant $\mathbf{P}$HYM metrics inducing h, then they must be asymptotic to the same $\mathbf{P}$HYM metric H_D on $\mathscr{E}_D$ (by uniqueness in the compact case). Then, for some $\delta > 0$,

$$s := \log(H_0^{-1}H) \in C^\infty_\delta(V\backslash S, i\mathfrak{su}(E, H_0)).$$

Moreover, by [Siu87, p. 13],

$$\Delta \log \operatorname{tr} e^s \leqslant 0$$

on $V\backslash S$. The argument in the proof of [BS94, Theorem 2(a)] shows that $\log \operatorname{tr} e^s \in W^{1,2}_{\mathrm{loc}}(V)$; hence, $\log \operatorname{tr} e^s$ is weakly subharmonic and thus $\log \operatorname{tr} e^s \leqslant \log \operatorname{rk} \mathscr{E}$ because s tends to zero at infinity. However, because of the inequality of arithmetic and geometric means, $\log \operatorname{tr} e^s \geqslant \log \operatorname{rk} \mathscr{E}$ with equality if and only if $s = 0$. □

A Useful formulae for Chern connections

Let $\mathscr{E} = (E, \bar{\partial})$ be a rank r holomorphic vector bundle. Given a Hermitian metric H on $\mathscr{E}$, there exists a unique Hermitian covariant derivative $\nabla = \nabla_H$ on E such that $\nabla_H^{0,1} = \bar{\partial}$. The connection A_H associated with ∇_H is called the **Chern connection** induced by H.

Fix a Hermitian metric H_0 and $s \in \Gamma(i\mathfrak{u}(E, H_0))$. Set

$$H := H_0 e^s \quad \text{and} \quad \tilde{A}_s := e_*^{s/2} A_H = e_*^{s/2} A_{H_0 e^s}.$$

Since $e^{s/2}\colon\ (E, H) \to (E, H_0)$ is an isometry, both $\tilde{A}_0 = A_{H_0}$ and $\tilde{A}_s$ are connections on the principal $\mathrm{U}(r)$–bundle $\mathrm{U}(E, H_0)$. Set

$$\mathfrak{K}(s) := \mathrm{Ad}(e^{s/2}) K_{H_0 e^s} = i\Lambda F_{\tilde{A}_s}.$$

All of the following results can be found in [LT06, Section 6], in the setting of holomorphic principal bundles. We summarise them here for the reader's convenience.

Proposition A.1. *We have*

$$\begin{aligned}\mathfrak{K}(s) = &(2 - 2\cosh(\mathrm{ad}_{s/2})) K_{H_0} \\ &+ \frac{1}{2}\Theta(s) \nabla_{H_0}^* \nabla_{H_0} s \\ &+ \frac{i}{2}\Lambda\left(\bar{\partial}\Upsilon(-s/2) \wedge \partial_{H_0} s\right) - \frac{i}{2}\Lambda\left(\partial_{H_0}\Upsilon(s/2) \wedge \bar{\partial} s\right) \\ &- \frac{i}{4}\Lambda\left(\Upsilon(-s/2)\partial_{H_0} s \wedge \Upsilon(s/2)\bar{\partial} s + \Upsilon(s/2)\bar{\partial} s \wedge \Upsilon(-s/2)\partial_{H_0} s\right)\end{aligned}$$

with $\Upsilon(s) \in \mathrm{End}(\mathfrak{gl}(E))$ *defined by*

$$\Upsilon(s) := \frac{e^{\mathrm{ad}_s} - \mathrm{id}}{\mathrm{ad}_s} \tag{A.2}$$

and $\Theta(s) \in \mathrm{End}(\mathfrak{gl}(E))$ *defined by*

$$\Theta(s) := \frac{\Upsilon(s/2) + \Upsilon(-s/2)}{2}. \tag{A.3}$$

Remark A.4. Since $\mathrm{ad}_s := [s, \cdot] \in \Gamma(\mathrm{End}(\mathfrak{gl}(E)))$ is self-adjoint with respect to H_0, so is $\Upsilon(s)$. Both $\cosh(\mathrm{ad}_{s/2})$ and $\Theta(s)$ preserve $\mathfrak{u}(E, H_0)$ because their power series expansions involve only even powers of ad_s and ad_s^2 preserves $\mathfrak{u}(E, H_0)$. Also note that $\Theta(s)$ is self-adjoint with respect to H_0 and its first eigenvalue is at least one.

Proof of Proposition A.1. Since $\partial_{H_0 e^s} = \partial_{H_0} + e^{-s}\partial_{H_0} e^s$, we have

$$\begin{aligned}\partial_{\tilde{A}_s} &= e^{s/2}(\partial_{H_0} + e^{-s}(\partial_{H_0} e^s))e^{-s/2} \\ &= \partial_{H_0} + e^{s/2}(\partial_{H_0} e^{-s/2}) + e^{-s/2}(\partial_{H_0} e^s)e^{-s/2} \\ &= \partial_{H_0} + e^{-s/2}(\partial_{H_0} e^{s/2})\end{aligned} \tag{A.5}$$

using $\partial e^{s/2} = \partial(e^s e^{-s/2}) = e^s \partial e^{-s/2} + (\partial e^s)e^{-s/2}$, and

$$\bar{\partial}_{\tilde{A}_s} = e^{s/2}\bar{\partial}e^{-s/2} = \bar{\partial} + e^{s/2}(\bar{\partial}e^{-s/2}) = \bar{\partial} - (\bar{\partial}e^{s/2})e^{-s/2}. \tag{A.6}$$

Using

$$\mathrm{d}_x \exp(y) = (\Upsilon(x)y)e^x = e^x(\Upsilon(-x)y) \tag{A.7}$$

we obtain

$$\tilde{A}_s = \tilde{A}_0 + \frac{1}{2}\Upsilon(-s/2)\partial_{H_0} s - \frac{1}{2}\Upsilon(s/2)\bar{\partial}s.$$

From this it follows that

$$\begin{aligned}F_{\tilde{A}_s} = F_{H_0} &+ \frac{1}{2}\Upsilon(-s/2)\bar{\partial}\partial_{H_0} s - \frac{1}{2}\Upsilon(s/2)\partial_{H_0}\bar{\partial}s \\ &+ \frac{1}{2}\bar{\partial}\Upsilon(-s/2) \wedge \partial_{H_0} s - \frac{1}{2}\partial_{H_0}\Upsilon(s/2) \wedge \bar{\partial}s \\ &- \frac{1}{4}\left(\Upsilon(-s/2)\partial_{H_0} s \wedge \Upsilon(s/2)\bar{\partial}s + \Upsilon(s/2)\bar{\partial}s \wedge \Upsilon(-s/2)\partial_{H_0} s\right).\end{aligned}$$

Applying $i\Lambda$ and using the Kähler identities

$$i[\Lambda, \bar{\partial}] = \partial_{H_0}^* \quad \text{and} \quad i[\Lambda, \partial_{H_0}] = -\bar{\partial}^*$$

as well as

$$\partial_{H_0}^*\partial_{H_0} = \frac{1}{2}\nabla_{H_0}^*\nabla_{H_0} + \frac{1}{2}[K_{H_0}, \cdot] \quad \text{and} \quad \bar{\partial}^*\bar{\partial} = \frac{1}{2}\nabla_{H_0}^*\nabla_{H_0} - \frac{1}{2}[K_{H_0}, \cdot],$$

we obtain

$$\begin{aligned}e^{s/2}K_{H_0 e^s} = K_{H_0} &+ \frac{1}{4}(\Upsilon(-s/2) - \Upsilon(s/2))\,\mathrm{ad}_s\, K_{H_0} \\ &+ \frac{1}{4}(\Upsilon(s/2) + \Upsilon(-s/2))\nabla_{H_0}^*\nabla_{H_0} s \\ &+ \frac{i}{2}\Lambda\left(\bar{\partial}\Upsilon(-s/2) \wedge \partial_{H_0} s\right) - \frac{i}{2}\Lambda\left(\partial_{H_0}\Upsilon(s/2) \wedge \bar{\partial}s\right) \\ &- \frac{i}{4}\Lambda\left(\Upsilon(-s/2)\partial_{H_0} s \wedge \Upsilon(s/2)\bar{\partial}s + \Upsilon(s/2)\bar{\partial}s \wedge \Upsilon(-s/2)\partial_{H_0} s\right).\end{aligned}$$

This implies the asserted identity because

$$1 - \frac{x}{4}\left(\frac{e^{-x/2} - 1}{x/2} + \frac{e^{x/2} - 1}{x/2}\right) = 2 - 2\cosh(x/2). \qquad \square$$

Proposition A.8. *We have*

$$\mathrm{d}_s\mathfrak{K}(\hat{s}) = \frac{1}{4}\nabla^*_{\tilde{A}_s}\nabla_{\tilde{A}_s}(\mathrm{id} + \mathrm{Ad}(e^{-s/2}))\Upsilon(s/2)\hat{s} - \frac{1}{4}\Big[\mathfrak{K}(s), (\mathrm{id} - \mathrm{Ad}(e^{-s/2})\Upsilon(s/2)\hat{s}\Big].$$

Proof. We have

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} F_{\tilde{A}_{s+t\hat{s}}} = \mathrm{d}_{\tilde{A}_s}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\tilde{A}_{s+\hat{t}s}\right).$$

Using (A.5) and (A.7), we compute

$$\begin{aligned}
&\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} e^{-(s+t\hat{s})/2}(\partial_{H_0}e^{(s+t\hat{s})/2})\\
&\quad= \frac{1}{2}\left(e^{-s/2}\partial_{H_0}\left(e^{s/2}\,\mathrm{Ad}(e^{-s/2})\Upsilon(s/2)\hat{s}\right) - (\mathrm{Ad}(e^{-s/2})\Upsilon(s/2)\hat{s})e^{-s/2}\partial_{H_0}e^{s/2}\right)\\
&\quad= \frac{1}{2}\left(\partial_{H_0}\left(\mathrm{Ad}(e^{-s/2})\Upsilon(s/2)\hat{s}\right) + \left[e^{-s/2}\partial_{H_0}e^{s/2}, \mathrm{Ad}(e^{-s/2})\Upsilon(s/2)\hat{s}\right]\right)\\
&\quad= \frac{1}{2}\partial_{\tilde{A}_s}\,\mathrm{Ad}(e^{-s/2})\Upsilon(s/2)\hat{s};
\end{aligned}$$

and, using (A.6) and (A.7), we compute

$$\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\bar{\partial}_{\tilde{A}_{s+t\hat{s}}} &= -\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}(\bar{\partial}e^{(s+t\hat{s})/2})e^{-(s+t\hat{s})/2}\\
&= -\frac{1}{2}\left(\bar{\partial}\left((\Upsilon(s/2)\hat{s})e^{s/2}\right)e^{-s/2} - (\bar{\partial}e^{s/2})e^{-s/2}(\Upsilon(s/2)\hat{s})\right)\\
&= -\frac{1}{2}\left(\bar{\partial}(\Upsilon(s/2)\hat{s}) - \left[(\bar{\partial}e^{s/2})e^{-s/2}, \Upsilon(s/2)\hat{s}\right]\right)\\
&= -\frac{1}{2}\bar{\partial}_{\tilde{A}_s}\Upsilon(s/2)\hat{s}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\mathfrak{K}(s+t\hat{s}) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} i\Lambda F_{\tilde{A}_{s+t\hat{s}}}\\
&= \frac{i}{2}\Lambda\left(\bar{\partial}_{\tilde{A}_s}\partial_{\tilde{A}_s}\,\mathrm{Ad}(e^{-s/2})\Upsilon(s/2)\hat{s} - \bar{\partial}_{\tilde{A}_s}\bar{\partial}_{\tilde{A}_s}\Upsilon(s/2)\hat{s}\right)\\
&= \frac{1}{4}i\Lambda(\bar{\partial}_{\tilde{A}_s}\partial_{\tilde{A}_s} - \partial_{\tilde{A}_s}\bar{\partial}_{\tilde{A}_s})(\mathrm{id} + \mathrm{Ad}(e^{-s/2}))\Upsilon(s/2)\hat{s}\\
&\qquad - \frac{1}{4}i\Lambda(\bar{\partial}_{\tilde{A}_s}\partial_{\tilde{A}_s} + \partial_{\tilde{A}_s}\bar{\partial}_{\tilde{A}_s})(\mathrm{id} - \mathrm{Ad}(e^{-s/2}))\Upsilon(s/2)\hat{s}\\
&= \frac{1}{4}\nabla^*_{\tilde{A}_s}\nabla_{\tilde{A}_s}(\mathrm{id} + \mathrm{Ad}(e^{-s/2}))\Upsilon(s/2)\hat{s}\\
&\qquad - \frac{1}{4}\Big[\mathfrak{K}(s), (\mathrm{id} - \mathrm{Ad}(e^{-s/2})\Upsilon(s/2)\hat{s}\Big].
\end{aligned}$$

□

Proposition A.9. *We have*

$$\langle \mathfrak{K}(s) - K_{H_0}, s\rangle = \langle i\Lambda\bar\partial(e^{-s}\partial_{H_0}e^s), s\rangle = \frac{1}{4}\Delta|s|^2 + \frac{1}{2}|\upsilon(-s)\nabla_{H_0}s|^2 \tag{A.10}$$

where $\upsilon(s) \in \mathrm{End}(\mathfrak{gl}(E))$ *is defined by*

$$\upsilon(s) \coloneqq \sqrt{\Upsilon(s)}. \tag{A.11}$$

Proof. We compute

$$\begin{aligned}\langle i\Lambda\bar\partial(e^{-s}\partial_{H_0}e^s), s\rangle &= \langle i\Lambda\bar\partial(\Upsilon(-s)\partial_{H_0}s), s\rangle\\ &= i\Lambda\bar\partial\langle\Upsilon(-s)\partial_{H_0}s, s\rangle + i\Lambda\langle\Upsilon(-s)\partial_{H_0}s \wedge \partial_{H_0}s\rangle\\ &= \partial^*\langle\partial_{H_0}s, \Upsilon(s)s\rangle + \langle\Upsilon(-s)\partial_{H_0}s, \partial_{H_0}s\rangle\\ &= \partial^*\langle\partial_{H_0}s, s\rangle + \left|\upsilon(-s)\partial_{H_0}s\right|^2\\ &= \frac{1}{2}\partial^*\partial|s|^2 + |\upsilon(-s)\partial_{H_0}s|^2.\end{aligned}$$

□

B The Donaldson functional

Let (X, g, I) be a compact Kähler manifold and let $\mathscr{E}$ be a holomorphic vector bundle over X. Given a metric H_0 and $s \in C^\infty(X, i\mathfrak{su}(\mathscr{E}, H_0))$, the value of the Donaldson functional at (H_0, H_0e^s) is

$$\mathscr{M}(H_0, H_0e^s) \coloneqq \int_0^1 \int_X \langle s, \mathrm{Ad}(e^{us/2})K_{H_0e^{us}}\rangle\, \mathrm{d}u.$$

This functional was introduced in [Don85, Section 1.2] and [Don87, §II]. We refrain from a lengthy discussion and only marshal the following three facts, which are used in Section 5.

Proposition B.1 ([Sim88, Proposition 5.1]). *We have*

$$\mathscr{M}(H_0, H_2) = \mathscr{M}(H_0, H_1) + \mathscr{M}(H_1, H_2).$$

Proposition B.2. *We have* $\mathscr{M}(H_0, H_0e^s) \lesssim \int_X |s||K_{H_0e^s}|$.

Proof. This holds because $m(u) \coloneqq \mathscr{M}(H_0, H_0e^{us})$ is convex [Don87, Proof of Lemma 24], $m(0) = 0$ and $m'(1) \lesssim \int_X |s||K_{H_0e^s}|$. □

Theorem B.3 (Donaldson [Don87, Lemma 24]; see also [Sim88, Proposition 5.3]). *If* H_0 *is* **P***HYM, then*

$$\|s\|_{L^2} - 1 \lesssim \mathscr{M}(H_0, H_0e^s).$$

C Bando–Siu interior estimate

Theorem C.1 (Bando and Siu [BS94, Proposition 1]). *Let (X, g, I) be a Kähler manifold of dimension n with bounded geometry and let $\mathscr{E}$ be a holomorphic vector bundle over X. If H_0 and H are Hermitian metrics on $\mathscr{E}$ and $s := \log(H_0^{-1}H) \in C^\infty(X, i\mathfrak{su}(\mathscr{E}, H_0))$, then*

$$\begin{aligned} r^{k+2-\frac{2n}{p}} \|\nabla_{H_0}^{k+2} s\|_{L^p(B_r(x))} &\leqslant \varepsilon_{k,p}\Big(\|s\|_{L^\infty(B_{2r}(x))} + \|K_H\|_{L^\infty(B_{2r}(x))} + r^{k-\frac{2n}{p}}\|\nabla^k K_H\|_{L^p(B_{2r}(x))} \\ &\qquad + \sum_{j=0}^{k} r^{2+i}\|\nabla_{H_0}^i F_{H_0}\|_{L^\infty(B_{2r}(x))}\Big) \end{aligned}$$

where $\varepsilon_{k,p}$ is a smooth function which vanishes at the origin and depends only on $k \in \mathbf{N}$, $p \in (1, \infty)$, and the geometry of X.

It suffices to prove this in the case where H_0 is a flat metric on a trivial holomorphic bundle over $\bar{B}_2 \subset \mathbf{C}^n$. The theorem is not a straight-forward consequence of standard bootstrapping techniques because we only have

$$\Delta s = A(K_H) + C(\nabla s \otimes \nabla s)$$

where A and C are linear with coefficients depending on s; see Proposition A.1. The usual Sobolev estimates will not suffice to prove Theorem C.1 without any control of ∇s. However, if we assume $C^{0,\beta}$ bounds on ∇s of the above form, then the usual method does give the desired estimates. It is well known to analysts that for an equation of this form $C^{0,\beta}$ bounds on ∇s can be obtained from a bound on the Morrey norm $\|\nabla s\|_{L^{2,2n-2+2\alpha}}$; see Definition E.1. We give full details of this fact, which is completely general and has nothing to do with Hermitian Yang–Mills metrics, in Appendix D. All of this being said, it thus suffices to prove the following proposition.

Proposition C.2. *Denote by H_0 a flat Hermitian metric on the trivial holomorphic bundle of rank r over $\bar{B}_2 \subset \mathbf{C}^n$. If $H = H_0 e^s$ with $s \in C^\infty(\bar{B}_2, i\mathfrak{su}(r))$, then*

$$\begin{aligned} [s]_{C^{0,\alpha}(\bar{B}_1)} &\lesssim \|\nabla s\|_{L^{2,2n-2+2\alpha}(B_1)} \\ &\leqslant \varepsilon(\|s\|_{L^\infty(B_2)} + \|K_H\|_{L^\infty(B_2)}) \end{aligned}$$

where $\alpha \in (0, 1)$ depends on $\|s\|_{L^\infty(B_2)}$ in a monotonely decreasing way, and ε is a smooth function which vanishes at the origin.

Proof. For $x \in B_1$ define $f_x \colon (0, 1] \to [0, \infty)$ by

$$f_x(r) := \int_{B_r(x)} G_x |\nabla s|^2$$

with $G_x(\cdot) \coloneqq |\cdot - x|^{2-2n}$. We will show that

$$f_x(r) \leqslant \varepsilon r^{2\alpha}$$

with ε and α as in the proposition. This implies the asserted Morrey bound.

In the following we fix $x \in B_1$ and $r \in (0, 1/2]$ and omit writing the subscript x to simplify notation.

Step 1. *We have $f(r) \leqslant \varepsilon$.*

Fix a smooth function $\chi\colon [0, \infty) \to [0, 1]$ which is equal to one on $[0, 1]$ and vanishes outside $[0, 2]$. Set $\chi_r(\cdot) \coloneqq \chi(|\cdot - x|/r)$. Using

$$|\nabla s|^2 \lesssim \varepsilon \cdot (1 - \Delta |s|^2),$$

which follows from Proposition A.9 and the observation before (5.5), we compute

$$\begin{aligned} f(r) &\leqslant \int_{B_{2r}(x)} \chi_r G \cdot |\nabla s|^2 \\ &\lesssim \varepsilon \int_{B_{2r}(x)} \chi_r G \cdot (-\Delta |s|^2) + \chi_r G \\ &\lesssim \varepsilon r^{-2n} \int_{B_{2r}(x) \setminus B_r(x)} |s|^2 + \varepsilon r^2 \\ &\leqslant \varepsilon. \end{aligned}$$

Step 2. *We have $f(r) \leqslant \gamma f(2r) + \varepsilon r^2$ for some constant $\gamma \in (0, 1)$ depending on $\|s\|_{L^\infty(B_2)}$.*

Set

$$\bar{s} \coloneqq \fint_{B_{2r}(x) \setminus B_r(x)} s \in i\mathfrak{su}(r) \quad \text{and} \quad \sigma \coloneqq \log(e^{-\bar{s}} e^s).$$

Observe that

$$|\nabla s|^2 \lesssim M |\nabla \sigma|^2 \quad \text{and} \quad |\sigma|^2 \lesssim M |s - \bar{s}|^2$$

with $M > 0$ some constant depending on $\|s\|_{L^\infty(B_2)}$ and $\|K_H\|_{L^\infty(B_2)}$ in a monotonely increasing way. Arguing as in the previous step we have

$$|\nabla \sigma|^2 \leqslant M(1 - \Delta |\sigma|^2).$$

Using the above and Poincaré's inequality we have

$$\begin{aligned}
\int_{B_r(x)} G|\nabla s|^2 &\lesssim M \int_{B_{2r}(x)} \chi_r G \cdot (-\Delta|\sigma|^2) + \varepsilon \chi_r G \\
&\lesssim M \cdot r^{-2n} \int_{B_{2r}(x)\setminus B_r(x)} |\sigma|^2 + \varepsilon r^2 \\
&\lesssim M^2 \cdot r^{-2n} \int_{B_{2r}(x)\setminus B_r(x)} |s-\bar{s}|^2 + \varepsilon r^2 \\
&\lesssim M^2 \cdot r^{2-2n} \int_{B_{2r}(x)\setminus B_r(x)} |\nabla s|^2 + \varepsilon r^2 \\
&\lesssim M^2 \int_{B_{2r}(x)\setminus B_r(x)} G|\nabla s|^2 + \varepsilon r^2.
\end{aligned}$$

This gives the asserted inequality with $\gamma = M^2/(1+M^2)$.

Step 3. *We have $f(r) \leqslant \varepsilon r^{2\alpha}$.*

We can assume that $\gamma \geqslant 1/2$. Set $g(r) := f(r) + \frac{\varepsilon r^2}{4\gamma - 1}$. By the second step

$$g(r) \leqslant \gamma^k g(2^k r).$$

Setting $k := \log_2\lceil 1/2r \rceil$, we have $\gamma^k \lesssim r^{2\alpha}$ for some $\alpha \in (0,1)$ depending only on γ; hence, by the first step

$$f(r) \leqslant \varepsilon r^{2\alpha}. \qquad \square$$

D Hildebrandt's $C^{1,\beta}$ estimate

The following result is well-known to analysts. It can be traced back to Hildebrandt's work on harmonic maps [Hil85, Section 6].

Proposition D.1. *Suppose $\alpha \in (0,1)$. Let U be an open subset of $\mathbf{R}^n$ with smooth boundary, and let $f\colon \bar{U} \to \mathbf{R}^k$ be a solution of a partial differential equation of the form*

$$\Delta f = A + B(\nabla f) + C(\nabla f \otimes \nabla f) \tag{D.2}$$

where $A \in C^0(\bar{U}, \mathbf{R}^k)$, $B \in C^0(\bar{U}, \mathrm{End}(\mathbf{R}^k))$, and $C \in C^0(\bar{U}, \mathrm{Hom}(\mathbf{R}^k \otimes \mathbf{R}^k, \mathbf{R}^k))$. For each $V \subset\subset U$, we have

$$\|\nabla f\|_{C^{0,\beta}(V)} \leqslant \varepsilon\left(\|\nabla f\|_{L^{n-2+2\alpha,2}(U)}\right)$$

where ε is a smooth increasing function vanishing at the origin (depending on A, B, C, U and V), and $\beta \in (0,1)$ depends only on α.

We will make heavy use of Morrey and Campanato spaces. For the reader's convenience all necessary definitions and results are summarised in Appendix E.

Proof. Set $R \coloneqq d(V, \partial U)$. Define $\phi\colon [0, R] \to [0, \infty)$ by

$$\phi(r) \coloneqq \sup\left\{\int_{B_r(x)} |\nabla f - \overline{\nabla f}_{x,r}|^2 : x \in V\right\}.$$

By definition

$$[\nabla f]_{\mathcal{L}^{2,\lambda}(V)} \leqslant \sup\left\{r^{-\lambda}\phi(r) : r > 0\right\} \leqslant [\nabla f]_{\mathcal{L}^{2,\lambda}(U)}.$$

We will show that

$$\phi(r) \leqslant \varepsilon r^{n+2\beta}$$

with ε as in the proposition. The assertion then follows from Morrey's Embedding Theorem in the form of Theorem E.5.

Trivially, we have

$$\phi(r) \leqslant \varepsilon r^{n-2+2\alpha}.$$

The following proposition strengthens this estimate using (D.2).

Proposition D.3. *For* $0 < s \leqslant r \leqslant R$ *and* $\alpha \leqslant 1$*, we have*

$$\phi(s) \leqslant c\left(\frac{s}{r}\right)^{n+2}\phi(r) + \varepsilon r^{n-2+3\alpha}.$$

We will postpone the proof of Proposition D.3 while we explain how the proof of Proposition D.1 is completed. To improve the exponent we use the following lemma, whose proof is very simple and deferred to the end of this section.

Lemma D.4. *If* $\phi\colon [0, R] \to [0, \infty)$ *is a non-decreasing function and* $c, \varepsilon > 0$*,* $\alpha > \beta > 0$ *are constants such that for all* $0 < s \leqslant r \leqslant R$

$$\phi(s) \leqslant c\left(\frac{s}{r}\right)^{\alpha}\phi(r) + \varepsilon r^{\beta},$$

then we have

$$\phi(r) \lesssim_{c,\alpha,\beta} \left(\frac{\phi(R)}{R^\beta} + \varepsilon\right) r^\beta.$$

We derive that

$$\|\nabla f\|_{\mathcal{L}^{2,n-2+2\alpha'}(V)} \leqslant \varepsilon$$

with $\alpha' = \frac{3}{2}\alpha$. If $\alpha' < 1$, then by Proposition E.3 we have

$$\|\nabla f\|_{L^{2,n-2+2\alpha'}(V)} \leqslant \varepsilon$$

and we can restart the argument with α' instead of α and V instead of U. Iterating this a finite number of times we will eventually end up in the case $\alpha' > 1$. In this case

$$\phi(r) \leqslant \varepsilon r^{n+2\beta}$$

with $\beta = \frac{\alpha'-1}{2}$. This completes the proof of Proposition D.1. ∎

Proof of Proposition D.3. Fix a ball $B_r(x) \subset U$ with center $x \in V$. We may assume that $f(x) = 0$, because in all that follows we can work with $f - f(x)$ instead.

Step 1. *We can write $f = g + h$ with $g, h\colon \bar{B}_r(x) \to \mathbf{R}^k$ satisfying*

$$\Delta g = A + B(\nabla f) + C(\nabla f \otimes \nabla f) \quad \textit{and} \quad g|_{\partial B_r(x)} = 0 \tag{D.5}$$

and

$$\Delta h = 0 \quad \textit{and} \quad h|_{\partial B_r(x)} = f|_{\partial B_r(x)}.$$

Step 2. *We have*

$$\|g\|_{L^\infty(B_r(x))} \leqslant \varepsilon r^\alpha \quad \textit{and} \quad \|h\|_{L^\infty(B_r(x))} \leqslant \varepsilon r^\alpha.$$

By Theorem E.4 and Theorem E.5 we have $[f]_{C^{0,\alpha}(U)} \leqslant \varepsilon$. From $f(x) = 0$ it follows that $\|f\|_{L^\infty(B_r(x))} \leqslant \varepsilon r^\alpha$. The maximum principle implies the asserted bound on h; the bound on g then follows.

Step 3. *We have*

$$\int_{B_r(x)} |\nabla g|^2 \leqslant \varepsilon r^{n-2+3\alpha}.$$

Since g vanishes on $\partial B_r(x)$ and using (D.5),

$$\begin{aligned}
\int_{B_r(x)} |\nabla g|^2 &= \int_{B_r(x)} \langle \Delta g, g\rangle \\
&\lesssim \int_{B_r(x)} |g|(1 + |\nabla f|^2) \\
&\leqslant \varepsilon r^{n-2+3\alpha}.
\end{aligned}$$

Step 4. *For $s \leqslant r$, we have*

$$\int_{B_s(x)} |\nabla h - \overline{\nabla h}_{x,s}|^2 \lesssim \left(\frac{s}{r}\right)^{(n+2)} \int_{B_r} |\nabla h - \overline{\nabla h}_{x,r}|^2.$$

This is Theorem E.6 for ∇h.

Step 5. *We complete the proof of the proposition.*

Using the preceding steps, we compute

$$\begin{aligned}
\int_{B_s(x)} |\nabla f - \overline{\nabla f}_{x,s}|^2 &\leqslant \int_{B_s(x)} |\nabla h - \overline{\nabla h}_{x,s} + \nabla g - \overline{\nabla g}_{x,s}|^2 \\
&\lesssim \int_{B_s(x)} |\nabla h - \overline{\nabla h}_{x,s}|^2 + \int_{B_s(x)} |\nabla g|^2 \\
&\lesssim \left(\frac{s}{r}\right)^{n+2} \int_{B_r(x)} |\nabla h - \overline{\nabla h}_{x,r}|^2 + \int_{B_r(x)} |\nabla g|^2 \\
&\lesssim \left(\frac{s}{r}\right)^{n+2} \int_{B_r(x)} |\nabla f - \overline{\nabla f}_{x,r}|^2 + \varepsilon r^{n-2+3\alpha}.
\end{aligned}$$

Taking the supremum over $x \in V$ yields the asserted statement. □

Proof of Lemma D.4. This is similar to but somewhat simpler than [HL11, Lemma 3.4]. If we choose $\tau < 1$ such that $\gamma := c\tau^{\alpha-\beta} < 1$, then

$$\begin{aligned}\phi(\tau^k R) &\leqslant \gamma\phi(\tau^{k-1}R)\tau^\beta + \frac{\varepsilon}{\tau^\beta}(\tau^k R)^\beta \\ &\leqslant \left(\gamma^k \frac{\phi(R)}{R^\beta} + \frac{\varepsilon}{(1-\gamma)\tau^\beta}\right)(\tau^k R)^\beta.\end{aligned}$$

From this the assertion follows immediately. □

E Morrey and Campanato spaces

An excellent exposition of Morrey and Campanato spaces can be found in Struwe's lecture notes [Str14, Kapitel 8 and 10]. We only state the definitions and the results we make use of.

Assume that $U \subset \mathbf{R}^n$ is open with smooth boundary. Let $1 \leqslant p < \infty$ and $\lambda \geqslant 0$.

Definition E.1. The **Morrey space** $(L^{p,\lambda}(U), \|\cdot\|_{L^{p,\lambda}(U)})$ is the normed vector space defined by

$$L^{p,\lambda}(U) := \left\{ f \in L^p(U) : \|f\|_{L^{p,\lambda}(U)} < \infty \right\}$$

with

$$\|f\|_{L^{p,\lambda}(U)} := \sup_{x\in U, r>0} \left(r^{-\lambda} \int_{B_r(x)\cap U} |f|^p \right)^{1/p}.$$

Definition E.2. The **Campanato space** $(\mathcal{L}^{p,\lambda}(U), \|\cdot\|_{\mathcal{L}^{p,\lambda}(U)})$ is the normed vector space defined by

$$\mathcal{L}^{p,\lambda}(U) := \left\{ f \in L^p(U) : [f]_{\mathcal{L}^{p,\lambda}(U)} < \infty \right\}$$

and

$$\|f\|_{\mathcal{L}^{p,\lambda}(U)} := \|f\|_{L^p(U)} + [f]_{\mathcal{L}^{p,n}(U)}.$$

Here the **Campanato semi-norm** is defined by

$$[f]_{\mathcal{L}^{p,\lambda}(U)} := \sup_{x\in U, r>0} \left(r^{-\lambda} \int_{B_r(x)\cap U} |f - \bar{f}_{x,r}|^p \right)^{1/p}$$

with

$$\bar{f}_{x,r} := \fint_{B_r(x)\cap U} f.$$

Both Morrey and Campanato spaces are Banach spaces. The following shed some more light on the relation between Morrey, Campanato, and Hölder spaces, and the Campanato regularity properties of harmonic functions.

Proposition E.3 ([Str14, Lemma 10.3.1]). *If $\lambda \leqslant n$, then for all $f \in \mathcal{L}^{p,\lambda}(U)$ we have*

$$\|f\|_{L^{p,\lambda}(U)} \lesssim \|f\|_{\mathcal{L}^{p,\lambda}(U)}.$$

Theorem E.4 (Poincaré inequality). *For all $f \in L^{p,\lambda}(U)$, we have*

$$[f]_{\mathcal{L}^{p,\lambda+p}(U)} \lesssim \|\nabla f\|_{L^{p,\lambda}(U)}.$$

Theorem E.5 (Morrey embedding [Str14, Satz 8.6.5]). *For all $f \in \mathcal{L}^{p,n+p\alpha}(U)$, we have*

$$[f]_{C^{0,\alpha}(\bar{U})} \lesssim [f]_{\mathcal{L}^{p,n+p\alpha}(U)}$$

Theorem E.6 ([Str14, Lemma 10.2.1] and [HL11, Lemma 3.10]). *If $f \in W^{1,2}(B_r(x))$ satisfies*

$$\Delta f = 0$$

and $0 < s < r$, then

$$\int_{B_s(x)} |f - \bar{f}_{x,s}|^2 \lesssim \left(\frac{s}{r}\right)^{(n+2)} \int_{B_r(x)} |f - \bar{f}_{x,r}|^2.$$

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